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# APPROXIMATION BY CONTINUED FRACTIONS

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ABSTRACT. Let $x$ be a real irrational number whose continued fraction has infinitely many partial quotients not less than $k$. A simple proof is given that the inequality $|x - p/q| < (k^2 + 4)^{-\frac{1}{2}} q^{-2}$ has infinitely many rational solutions $p/q$. The constant $(k^2 + 4)^{-\frac{1}{2}}$ is best possible.

Recently, J. H. E. Cohn [1] gave a simple proof of Hurwitz' theorem, that for any real irrational number $x$ there are infinitely many rational numbers $p/q$ such that $|x - p/q| < 5^{-\frac{1}{2}}q^{-2}$. In this note we make a small change in Cohn's proof and get a much stronger result.

For $k \geq 1$, let $F(k)$ be the set of all real numbers $x$ such that $0 \leq x \leq 1$ and the continued fraction for $x$ has no partial quotient greater than $k$. Let $F(0) = \varnothing$. Real numbers $x$ and $y$ are called equivalent if their continued fractions eventually coincide.

**Theorem.** *Let $k \geq 1$ and let $x$ be a real irrational number not equivalent to an element of $F(k - 1)$. Then there are infinitely many rational numbers $p/q$ such that*

$$|x - p/q| < 1/(k^2 + 4)^{\frac{1}{2}} q^2. \tag{1}$$

*The constant $(k^2 + 4)^{-\frac{1}{2}}$ is best possible.*

**Proof.** Let $p_n/q_n$ denote the $n$th convergent of the continued fraction $[a_0, a_1, a_2, \cdots]$ of $x$. Following Cohn, we let $\theta_n = q_n|q_n x - p_n|$ and $\phi_n = \min(\theta_{n-1}, \theta_n, \theta_{n+1})$ and obtain

$$\frac{\theta_n}{q_n^2} + \frac{\theta_{n+1}}{q_{n+1}^2} = \left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}},$$

and so

$$\frac{q_{n+1}}{q_n} = \frac{1 \pm (1 - 4\theta_n\theta_{n+1})^{\frac{1}{2}}}{2\theta_n}. \tag{2}$$

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Then

$$\frac{1+(1-4\theta_n\theta_{n+1})^{1/2}}{2\theta_n} \geq \frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} \geq a_{n+1} + \frac{2\theta_{n-1}}{1+(1-4\theta_{n-1}\theta_n)^{1/2}},$$

and

$$2\phi_n a_{n+1} \leq 2\theta_n a_{n+1} \leq (1-4\theta_n\theta_{n+1})^{1/2} + (1-4\theta_{n-1}\theta_n)^{1/2} \leq 2(1-4\phi_n^2)^{1/2},$$

or $\phi_n \leq (a_{n+1}^2 + 4)^{-1/2}$. If $\phi_n = (a_{n+1}^2 + 4)^{-1/2}$, then $\theta_n = \theta_{n+1} = \phi_n$. But, by (2), this is impossible. Therefore, $\phi_n < (a_{n+1}^2 + 4)^{-1/2}$ for all $n$.

If $x$ is not equivalent to an element of $F(k-1)$, then for infinitely many $n$ we have $a_{n+1} \geq k$, and so $\phi_n < (k^2+4)^{-1/2}$. This means that infinitely many convergents of the continued fraction for $x$ satisfy (1).

The continued fraction for $x = ((k^2+4)^{1/2} - k)/2$ is $[0, k, k, k, \ldots]$, and so $x$ is not equivalent to an element of $F(k-1)$. If $A < (k^2+4)^{-1/2}$, it follows by a standard argument (see [2, Theorem 194] in the case $k = 1$) that $|x - p/q| < Aq^{-2}$ has only finitely many rational solutions $p/q$, and so the constant $(k^2+4)^{-1/2}$ is best possible.

**Remark.** For $k = 1$, this theorem gives Hurwitz' theorem. The set $F(1)$ contains only the number $(5^{1/2} - 1)/2$, the golden mean, and so when $k = 2$ the theorem gives the well-known result that, for any irrational number $x$ not equivalent to the golden mean, the inequality $|x - p/q| < 8^{-1/2}q^{-2}$ has infinitely many rational solutions $p/q$.

## REFERENCES

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