Additive problems in combinatorial number theory

Melvyn B. Nathanson
Office of the Provost and
Vice President for Academic Affairs
Lehman College (CUNY)
Bronx, New York 10468

1. Introduction

Many important questions in combinatorial number theory arise from the classical problems in additive number theory. Central to additive number theory is the study of bases of finite order. If A is a set of nonnegative integers, the h-fold sumset of A, denoted hA, is the set of all sums of h elements of A, with repetitions allowed. If hA is the set **N** of all nonnegative integers, then A is called a basis of order h. If hA contains all sufficiently large integers, then A is called an asymptotic basis of order h. Much of classical additive number theory is the study of sumsets hA, where A is the set of squares (Lagrange's theorem), or the k-th powers (Waring's problem), or the polygonal numbers (Gauss's theorem for triangular numbers and Cauchy's theorem for polygonal numbers of any order), or the primes (Goldbach's conjecture). Nathanson [24] recently gave a short and simple proof of Cauchy's polygonal number theorem.

Shnirel'man [32] created a new field of research in additive number theory when he discovered a powerful combinatorial criterion that implies that a set A is a basis of order h for some h, and proved that {0,1} U {primes} is a basis. Using Shnirel'man's method, Nathanson [23] proved that any set containing a positive proportion of the prime numbers is a basis of order h. Much of the work in combinatorial number theory concerns general properties of the classical additive bases and of arbitrary bases of finite order. In this paper I shall describe recent results and some unsolved additive problems in combinatorial number theory.

2. Thin bases

Let A be a set of nonnegative integers. Define the <u>counting function</u> A(x) of the set A as the number of positive elements of A not exceeding x. If A is an asymptotic basis of order h, then an easy combinatorial argument shows that $A(x) > c\cdot x^{1/h}$ for some constant $c > 0$ and all x sufficiently large. An asymptotic basis A of order h is <u>thin</u> if $A(x) < c'x^{1/h}$ for some $c' > 0$ and all x sufficiently large. Thin bases exist. The first examples were constructed by Chatrovsky [2], Raikov [31], and Stöhr [33]. The best result is due to Cassels [1], who constructed, for each $h \geq 2$, a family of bases A of order h such that $A(x) \sim c\cdot x^{1/h}$ as x tends to infinity.

Problem 1. Do the classical sequences in additive number theory contain subsequences that are thin bases?

This is not yet known, but some surprising results have been obtained. Let A be a finite set of integers, and let $|A|$ denote the cardinality of A. If hA contains $\{0,1,2,\ldots,n\}$, then A is called a <u>basis of order h for n</u>. Clearly, if A is a basis of order h for n, then $|A| > n^{1/h}$. Choi, Erdös, and Nathanson [3] proved the following.

THEOREM 1. For every $n \geq 2$ there is a finite set A of squares such that A is a basis of order 4 for n, and

$$|A| < (4/\log 2)n^{1/3}\log n.$$

This was proved by means of an explicit construction. Note that there are $[n^{1/2}]+1$ squares up to n, and $|A|/n^{1/2}$ tends to zero.

Erdös and Nathanson [10] used probability methods to obtain the following result in the infinite case.

THEOREM 2. For every $\delta > 0$ there exists a set A of squares such that

(i) A is a basis of order 4,

(ii) If $n \neq 4^r(8k+7)$, then $n \in 3A$, and

(iii) $A(x) \sim c\cdot x^{(1/3)+\delta}$ for some $c > 0$.

Zöllner [38] combined the two results above to prove the following.

THEOREM 3. For every $\delta > 0$ there is an integer n_0 with the property that for any $n \geq n_0$ there is a finite set A of squares such that A is a basis of order 4 for n, and

$$|A| < n^{(1/4)+\delta}.$$

Problem 2. Does there exist $c > 0$ such that for $n \geq n_0$ there is a finite set A of squares such that A is a basis of order 4 for n, and

$$|A| < c \cdot n^{(1/4)} ?$$

Using probability methods, Zöllner and Wirsing independently obtained the following results on infinite sets of squares.

THEOREM 4 (Zöllner [37]). Let $h \geq 4$. For every $\delta > 0$ there exists a set A of squares such that A is a basis of order h and

$$A(x) < x^{(1/h)+\delta}$$

for x sufficiently large.

THEOREM 5 (Wirsing [36]). Let $h \geq 4$. There exists a set A of squares such that A is a basis of order h and

$$A(x) < c(x\log x)^{1/h}$$

for some $c > 0$ and all x sufficiently large.

Problem 3. Construct an explicit example of a set A of squares such that A is a basis of order 4 and $A(x)/x^{1/2}$ tends to zero. Note that this is considerably weaker that the non-constructive results stated in Theorems 2, 4, and 5.

Nathanson [21] has also obtained a thin variant of Waring's problem.

THEOREM 6. Let $k \geq 3$ and $s > s_0(k)$. Choose σ such that $1-(1/s) < \sigma < 1$. There exists a set A of nonnegative k-th powers such that A is a basis of order s, and

$$A(x) \sim c \cdot x^{\sigma/k}$$

for some constant $c > 0$.

The proof requires probabilistic arguments as well as the Hardy-Littlewood asymptotic formula for the number of representations of an integer as the sum of s k-th powers.

Problem 4. Construct an explicit example of a set A of k-th powers such that, for some s, the set A is a basis of order s and $A(x)/x^{1/k}$ tends to zero.

There is a finite version of Theorem 6. Let f(n,k,s) denote the cardinality of the smallest finite set A of k-th powers such that A is a basis of order s for n. Clearly, $f(n,k,s) > n^{1/s}$. Define

$$\beta(k,s) = \limsup_{n \to \infty} \log f(n,k,s)/\log n.$$

Let g(k) denote the smallest integer h such that the set of all nonnegative k-th powers is a basis of order h. Nathanson [22] proved the following.

THEOREM 7. For $k \geq 3$ and $s \geq g(k)$,

$$f(n,k,s) < 2(s-g(k)+1)\cdot n^{1/(s-g(k)+k)}.$$

In particular, $\beta(k,s) \sim 1/s$ as s tends to infinity.

Finally, Wirsing [36] proved the following beautiful result on sums of primes.

THEOREM 8. For $h \geq 3$, there is a set P of prime numbers such that

(i) $n \in hP$ for all $n \equiv h \pmod 2$ and n sufficiently large,

(ii) $P(x) < c\cdot(x\log x)^{1/h}$.

3. Minimal bases

Recall that A is an <u>asymptotic basis of order h</u> if hA contains all sufficiently large integers. The asymptotic basis A is <u>minimal</u> if A is an asymptotic basis of order h, but no proper subset of A is an asymptotic basis of order h. This means that for each element $a \in A$ there are infinitely many positive integers n, each of whose representations as a sum of h elements of A must include the integer a as a summand. Stöhr [34] first defined minimal asymptotic bases, and the earliest results were obtained by Erdös, Härtter, and Nathanson [4,7,17,20].

It is important to note that not every asymptotic basis of order h contains a subset that is a minimal asymptotic basis of order h. Stöhr [34], for example, observed that for $h \geq 2$ the set

$$A = \{1\} \cup \{ih \mid i = 0,1,2,\ldots\}$$

does not contain a minimal asymptotic basis of order h. For $h = 2$, Erdös and Nathanson [8] have constructed a set A with the property that, for any subset S of A, the set $A \setminus S$ is an asymptotic basis of order 2 if and only if S is finite. Since the infinite set A contains no maximal finite subset, it follows that A does not contain a minimal asymptotic basis of order 2.

Problem 5. Let $h \geq 3$. Construct a set A of nonnegative integers such that, for any subset S of A, the set $A \setminus S$ is an asymptotic basis of order h if and only if S is finite.

Erdös and Nathanson [9] have obtained a sufficient condition for an asymptotic basis of order 2 to contain a minimal asymptotic basis of order 2. For any set A of integers, let $r(n) = r_A(n)$ denote the number of solutions of the equation $n = a + a'$, where $a, a' \in A$ and $a \leq a'$.

THEOREM 9. Let A be an asymptotic basis of order 2. If $r(n) \geq c \cdot \log n$ for some constant $c > 1/\log(4/3)$ and all n sufficiently large, then A contains a minimal asymptotic basis of order 2.

Problem 6. Does the condition $r(n) > c \cdot \log n$ for <u>some</u> constant $c > 0$ and all n sufficiently large imply that A contains a minimal asymptotic basis of order 2?

In the opposite direction, Erdös and Nathanson [12] have proved the following.

THEOREM 10. Let $t \geq 1$. There exists an asymptotic basis A of order 2 such that $r(n) > t$ for all n sufficiently large, but A does not contain a minimal asymptotic basis of order 2.

Problem 7. Extend Theorems 9 and 10 to asymptotic bases of orders $h \geq 3$.

The following question seems to be very difficult.

Problem 8. If A is an asymptotic basis of order 2 such that r(n) tends to infinity, then must A contain a minimal asymptotic basis of order 2?

4. Fat minimal bases

Minimal bases are an extremal class of additive bases. I shall consider next an extremal property of this extremal class, namely, minimal asymptotic bases that are as "fat" or as "thin" as possible.

Recall that the counting function A(x) of the set A is the number of positive elements of A not exceeding x. Define the lower asymptotic density of A by $d_L(A) = \liminf A(x)/x$. If $\alpha = \lim A(x)/x$ exists, then α is called the asymptotic density of A, and denoted d(A).

Using Kneser's addition theorem [19] for the lower asymptotic density of sumsets, Nathanson and Sárközy [28] recently proved the following.

THEOREM 11. Let $h \geq 2$, and let A be an asymptotic basis of order h. If B is a subset of A such that $d_L(B) > 1/h$, then $A \backslash B$ contains a finite set F such that $B \cup F$ is an asymptotic basis of order h.

They drew two consequences from this result.

THEOREM 12. Let $h \geq 2$, and let A be an asymptotic basis of order h such that $d_L(A) = (1/h) + \delta$, where $\delta > 0$. Let $0 < \mu < \delta$. Then there is a subset W of A with asymptotic density $d(W) = \mu$ such that $A \backslash W$ is an asymptotic basis of order h.

THEOREM 13. Let $h \geq 2$. If A is a minimal asymptotic basis of order h, then $d_L(A) \leq 1/h$.

The next result shows that the estimate above is best possible.

THEOREM 14 (Erdös and Nathanson [11]). For every $h \geq 2$ there exist minimal asymptotic bases A of order h with $d(A) = 1/h$.

The proof of the theorem is by the somewhat complicated construction of explicit examples. The idea is as follows: Let A' be the union of sets B and C', where $B = \{b_i\}$ is a set of positive integers such that $b_i \equiv 1 \pmod h$ for all i, and b_{i+1}/b_i tends to infinity, and where C' consists of all nonnegative multiples of h. Then A' is an asymptotic basis of order h with $d(A') = 1/h$. One can construct inductively a subset C of C' such that $C' \backslash C$ has density 0 and $A = B \cup C$ is a minimal asymptotic basis of order h. Then $d(A) = 1/h$.

It is worth noting that the preceding theorems are among the few results about minimal asymptotic bases that hold for all $h \geq 2$, and not just $h = 2$.

Problem 9. Let $h \geq 2$. Does there exist a minimal asymptotic basis A of order h such that $A(x) = x/h + O(1)$?

Examination of the proof of Theorem 14 in the case $h = 2$ shows that it produces a minimal asymptotic basis $A = \{a_i\}$ of order 2 such that $a_{i+1} > a_i$ and $\limsup (a_{i+1} - a_i) = 4$. It is easy to show that there cannot exist a minimal asymptotic basis $A = \{a_i\}$ of order 2 with $\limsup (a_{i+1} - a_i) = 2$.

Problem 10. Does there exist a minimal asymptotic basis $A = \{a_i\}$ of order 2 with $\limsup (a_{i+1} - a_i) = 3$?

For $h = 2$, Erdös and Nathanson [11] have extended Theorem 14 in the following way.

THEOREM 15. For every $\alpha \in (0, 1/2]$ there exists a minimal asymptotic basis A of order 2 with asymptotic density $d(A) = \alpha$.

Problem 11. Let $h \geq 3$. Show that for every $\alpha \in (0, 1/h]$ there exists a minimal asymptotic basis A of order h with asymptotic density $d(A) = \alpha$.

5. Thin minimal bases

The results in the preceding section are about minimal asymptotic bases that are as "fat" as possible. I shall now consider the construction of "thin" minimal asymptotic bases.

Recall that an asymptotic basis A is thin if $A(x) < c'x^{1/h}$ for some $c' > 0$ and all x sufficiently large. Nathanson [20] constructed the first example of a thin minimal asymptotic basis of order 2. This construction has recently been generalized to produce thin minimal asymptotic bases of order h for every $h \geq 2$.

THEOREM 16 (Nathanson [26]). Let $h \geq 2$. There exists a minimal asymptotic basis A of order h such that

$$c \cdot x^{1/h} < A(x) < c'x^{1/h}$$

for positive constants c and c' and all x sufficiently large.

Jia and Nathanson [18] have improved this as follows.

THEOREM 17. Let $h \geq 2$ and let $\mu \in [1/h, 1)$. There exists a minimal asymptotic basis A of order h such that

$$c \cdot x^{\mu} < A(x) < c'x^{\mu}$$

for positive constants c and c' and all x sufficiently large.

The minimal asymptotic bases in Theorems 16 and 17 are all constructed by the following method: If W is a subset of the nonnegative integers **N**, let A(W) consist of all numbers of the form $\Sigma_{f \in F} 2^f$, where F is a finite, nonempty subset of W. Let

$$\mathbf{N} = W_0 \cup W_1 \cup \cdots \cup W_{h-1}$$

be a partition of **N** into pairwise disjoint, nonempty sets. Then the set

$$A = A(W_0) \cup A(W_1) \cup \cdots \cup A(W_{h-1}) \qquad (*)$$

is an asymptotic basis of order h. If the partition is chosen appropriately, the basis A is minimal.

Not every partition, however, gives rise to a minimal basis. For $h = 2$, Nathanson [26] gave an example of a partition $\mathbf{N} = W_0 \cup W_1$ such that the corresponding asymptotic basis $A = A(W_0) \cup A(W_1)$ is not a minimal asymptotic basis of order 2.

Problem 12. Let $h \geq 2$. Determine the partitions of **N** into h sets $\mathbf{N} = W_0 \cup W_1 \cup \cdots \cup W_{h-1}$ such that the set A defined by (*) is not a minimal asymptotic basis of order h.

6. A multiplicative variant of the Erdös-Turán conjecture

Let **N** denote the set of nonnegative integers. Let $h \geq 2$, and let A be an asymptotic basis of order h. Let $r_h(n)$ denote the number of representations of n as the sum of h elements of A. In 1941, Erdös and Turán [14] conjectured that if A is an asymptotic basis of order 2, then $\limsup r_2(n) = \infty$. This has not yet been proven. More generally, one can conjecture that if A is an asymptotic basis of order h, then $\limsup r_h(n) = \infty$. The Erdös-Turán conjecture can be restated as follows: Let A be a subset of **N**, and define $s = \liminf r_h(n)$ and $t = \limsup r_h(n)$. Then $s > 0$ implies that $t = \infty$.

Let $A_1,\ldots,A_h$ be subsets of **N**. Let $r'(n)$ denote the number of representations of n in the form $n = a_1 +\cdots+ a_h$, where $a_i \in A_i$ for $i = 1,\ldots,h$. Define $s' = \liminf r'(n)$ and $t' = \limsup r'(n)$. The sets $A_1,\ldots,A_h$ form an <u>additive system of order h</u> if $s' > 0$, that is, if $r'(n) > 0$ for all n sufficiently large.

Here is a simple example of an additive system of order h: For $m \geq 2$, let A_1 consist of all nonnegative multiples of m, let A_2 consist of exactly k complete sets of residues modulo m, and let $A_i = \{0\}$ for $i = 3,\ldots,h$. Then the sets $A_1,\ldots,A_h$ form an additive system of order h such that $s' = t' = k$. This example shows that the analogue of the Erdös-Turán conjecture does not hold for additive systems.

It is remarkable that the multiplicative version of the Erdös-Turán conjecture is true. Let **N*** denote the set of positive integers, and let B be a subset of **N***. Let $h \geq 2$. If every sufficiently large integer can be represented as a product of h elements of B, with repetitions allowed, then B is called a <u>multiplicative asymptotic basis of order h</u>. Let $g(n)$ denote the number of representations of n as a product of h elements of B. Using results from extremal graph theory, Erdös [5] obtained the following result.

THEOREM 18. Let $h \geq 2$. If B is a multiplicative asymptotic basis of order h, then $\limsup g(n) = \infty$.

Recently, Nešetřil and Rödl [30] used Ramsey's theorem to give a short proof of this result.

Let $B_1,\ldots,B_h$ be subsets of **N***. Let $g'(n)$ denote the number of representations of n in the form $n = b_1\cdots b_h$, where $b_i \in B_i$ for $i = 1,\ldots,h$. Define $s' = \liminf g'(n)$ and $t' = \limsup g'(n)$. The sets $B_1,\ldots,B_h$ form a <u>multiplicative system of order h</u> if $s' > 0$, that is, if $g'(n) > 0$ for all sufficiently large n.

Here is a simple example of a multiplicative system: Let $B_1 = \{1,2,4,8,\ldots\}$ be the set of powers of 2, let $B_2 = \{1,3,5,7,\ldots\}$ be the set of odd numbers, and let $B_i = \{1\}$ for $i = 3,\ldots,h$. Since every positive integer n has a unique representation as a product $n = b_1 \cdots b_h$ with $b_i \in B_i$, the sets $B_1,\ldots,B_h$ form a multiplicative system of order h with $s' = t' = 1$. Thus, $s' > 0$ does not imply that $t' = \infty$ for multiplicative systems.

Although this construction suggests that an analogue of the Erdös-Turán conjecture will not hold for multiplicative systems, the opposite, in fact, is true. Using a version of Ramsey's theorem, Nathanson [25] proved the following:

THEOREM 19. Let $B_1,\ldots,B_h$ be a multiplicative system of order h. If $s' = \liminf g'(n) \geq 2$, then $t' = \limsup g'(n) = \infty$.

Indeed, Nathanson [25] obtained the following more precise result.

THEOREM 20. For $h \geq 2$, let M(h) be the set of all pairs (s',t') such that $s' = \liminf g'(n)$ and $t' = \limsup g'(n)$ for some multiplicative system $B_1,\ldots,B_h$ of order h. Then

$$M(h) = \{(1,y) \mid y \in \mathbf{N}^*\} \cup \{(x,\infty) \mid x = 1,\ldots,h\}.$$

Note that Theorem 20 implies Theorems 18 and 19.

Problem 13. Can Ramsey theory be applied to the additive Erdös-Turán conjecture?

7. Finite sumsets containing special sets

Let n be a positive integer, and let A be a subset of $\{1,2,\ldots,n\}$. Denote the cardinality of A by $|A|$. Let $k \geq 3$ and $\delta > 0$. Szemerédi [35] proved that if $|A| > \delta n$ for $n \geq n(\delta,k)$, then A contains an arithmetic progression of length k. Nathanson and Sárközy [29] have obtained a lower bound for the length of the longest arithmetic progression contained in the h-fold sumset of a finite set.

THEOREM 21. Let N and k be positive integers. Let A be a subset of $\{1,2,\ldots,N\}$ such that

$$|A| \geq N/k + 1. \tag{3}$$

Then there exists an integer d with

$$1 \leq d \leq k-1 \tag{4}$$

such that if h and z are any positive integers satisfying the inequality

$$N/h + zd \leq |A| \tag{5}$$

then the sumset (2h)A contains an arithmetic progression with z terms and difference d.

Problem 14. Let $h \geq 2$ and let A be a "large" subset of $\{1,2,\ldots,n\}$. Find a good estimate for the length of the longest arithmetic progression contained in the sumset hA.

This problem is related to a question of Erdös and Freud [6]. They conjectured that if $A \subseteq \{1,2,\ldots,n\}$ and $|A| > n/3$, then there is a power of 2 that can be written as a sum of distinct elements of A. This is best possible, since, for $n = 3m$ and $A = \{3,6,9,\ldots,3m\}$, each sum of elements of A is divisible by 3, hence is not a power of 2. Using the method of trigonometric sums, Freiman [16] proved this conjecture. He showed that there is a constant $c > 0$ such that, for n sufficiently large, some power of 2 can be written as a sum of $c \cdot \log n$ distinct elements of A.

This result is not completely satisfactory, since the number of summands in Freiman's theorem tends to infinity as n tends to infinity. Does there exist an absolute constant h such that, for n sufficiently large, there is a power of 2 that can be represented as a sum of at most h distinct elements of A? Nathanson and Sárközy [29] have recently solved this problem. They used Theorem 21 to prove the following two results.

THEOREM 22. Let $m > 2^7 3^3 = 3,456$. If $A \subseteq \{1,2,\ldots,3m\}$ and $|A| \geq m+1$, then there is a power of 2 that can be written as the sum of at most 3,504 elements of A.

THEOREM 23. For m sufficiently large, if $A \subseteq \{1,2,\ldots,3m\}$ and $|A| \geq m+1$, then there is a power of 2 that can be written as the sum of at most 30,961 distinct elements of A.

Problem 15. Let h_1 (resp. h_1') be the least integer such that, for m sufficiently large, if A $\{1,2,\ldots,3m\}$ and $|A| > m$, then there is a power of 2 that can be written as a sum of exactly h_1 (resp. at most h_1') distinct elements of A. Determine the values of h_1 and h_1'.

8. Infinite sumsets containing special sets

Let A be an infinite set of integers. Szemerédi's theorem implies that if $d_U(A) > 0$, then A contains arbitrarily long finite arithmetic progressions. The set A, however, does not necessarily contain an infinite arithmetic progression. Indeed, it is easy to construct an example of a set A with $d_U(A) = 1$ such that neither A nor any sumset hA contains an infinite arithmetic progression. For real numbers x and y, let $[x,y]$ denote the set of all integers n such that $x \le n \le y$. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that t_{n+1}/t_n tends to infinity. Let

$$A = \bigcup_{n=1}^{\infty} [t_{2n} + 1,\ t_{2n+1}].$$

Then $d_U(A) = 1$, because

$$\begin{aligned} A(t_{2n+1})/t_{2n+1} &\ge (t_{2n+1} - t_{2n})/t_{2n+1} \\ &= 1 - t_{2n}/t_{2n+1} \to 1. \end{aligned}$$

Let $h \ge 1$. Since $hA \cap [ht_{2n-1} + 1,\ t_{2n}] = \emptyset$ for $n \ge n(h)$, the sumset hA contains arbitrarily long gaps, and so does not contain an infinite arithmetic progression.

Note that $d_L(A) = 0$ for the set A in the preceding example. If A is a set of nonnegative integers such that $d_L(A) > 0$, then some sumset hA must contain an infinite arithmetic progression. Denote by $h\hat{}A$ the set of all sums of h pairwise distinct elements of the set A. Erdös, Nathanson, and Sárközy [13] proved the following result.

THEOREM 24. Let A be a set of nonnegative integers such that $d_L(A) = \alpha \in (0,1/2]$. Let h be the smallest integer $\ge 1/\alpha$. Then

(i) $(h+1)\hat{}A$ contains an infinite arithmetic progression with difference $g \le h^2-h$, and

(ii) $(h^2-h)\hat{}A$ contains an infinite arithmetic progression with difference $g \le h+1$.

This result is best possible in the sense that for every $h \geq 1$ there exists a set A such that $d_L(A) = 1/h$, but the sumset hA does not contain an infinite arithmetic progression. For example, let $\{t_n\}$ be a strictly increasing sequence of positive integers such that t_{n+1}/t_n tends to infinity, and let the set A be the union of the integers in the intervals $[t_{n-1}, (t_n/h) - \sqrt{t_n}]$. Then $d_L(A) = 1/h$ and $d_U(A) = 1$. Since the sumset hA is disjoint from the interval $(t_n - h\sqrt{t_n}, t_n)$ for all $n \geq n(h)$, it follows that hA contains arbitrarily long gaps, and so hA does not contain an infinite arithmetic progression.

Erdös, Nathanson, and Sárközy [13] have also proved the following result, which is an infinite analogue of the Erdös-Freud problem.

THEOREM 25. Let B be a set of nonnegative integers such that $d_L(B) \geq 1/3$ and $3 \nmid b^*$ for some $b^* \in B$. Then infinitely many powers of 2 can be written as sums of either four or five distinct elements of B.

Problem 16. Let g_1 (resp. g_1') be the least integer such that, if A is any set of nonnegative integers with the properties that $d_L(A) \geq 1/3$ and $3 \nmid a$ for some $a \in A$, then some power of 2 can be written as the sum of exactly g_1 (resp. at most g_1') elements of A. Determine the precise values of g_1 and g_1'.

In response to Theorem 25, Erdös and Freud [6] have posed the following problem.

Problem 17. Let A be a set of positive integers such that $d_L(A) > 1/3$. Does the equation $a_i + a_j = 2^t$ have infinitely many solutions with $a_i, a_j \in A$? If so, this result would be best possible.

9. Sumsets containing k-free numbers

There is an analogous problem concerning square-free numbers. Erdös and Freud [6] asked: If $A \subseteq \{1,2,\ldots,4m\}$ and $|A| \geq m+1$, then is there a square-free number that can be written as a sum of distinct elements of A? The set $A = \{4,8,12,\ldots,4m\}$ shows that this would be best possible. Nathanson and Sárközy (unpublished) obtained the following result.

THEOREM 26. For m sufficiently large, if $A \subseteq \{1,2,\ldots,4m\}$ and $|A| \geq m+1$, then there are at least $O(\sqrt{n})$ square-free numbers, each of which can be written as a sum of either 20 or 21 distinct elements of the set A.

Using a clever combinatorial argument, Filaseta [15] has greatly improved this result.

THEOREM 27. Let $A \subseteq \{1,2,\ldots,4m\}$ be of maximal cardinality such that

(i) $A \not\subseteq \{4,8,12,\ldots,4m\}$,

(ii) $A \not\subseteq \{2,6,10,\ldots,4m-2\}$,

(iii) 2A contains no square-free number.

Then

$$2/9 \leq \liminf |A|/m \leq \limsup |A|/m \leq 4-32/\pi^2 = 0.757\ldots.$$

Filaseta has asked if $\lim_{m\to\infty} |A|/m$ exists.

Let Q_k denote the set of all k-free natural numbers, and let Q_k' denote the set of all odd, k-free numbers. The set Q_k has asymptotic density $1/\zeta(k)$, where $\zeta(k)$ is the Riemann zeta function, and Q_k' has asymptotic density $2^{k-1}/((2^k-1)\zeta(k))$.

Define the subset sum s(B) by $s(B) = \Sigma_{b\in B} b$. It is easy to find sets A such that $s(B) \notin Q_k$ for all subsets $B \subseteq A$. For example, let A be a set of multiples of d^k for some $d \geq 2$. Then $d^k | s(B)$ for all subsets B of A, and so $s(B) \notin Q_k$. Let $h \geq 2$. If we wish to consider only subset sums s(B) with $|B| = h$, then any set A, each of whose elements satisfies $a \equiv h^{k-1} \pmod{h^k}$, will have the property that $s(B) \notin Q_k$ whenever $B \subseteq A$ and $|B| = h$. In the case $h = 2$, if A is any subset of

$$\{n \geq 1 \mid n \equiv 2^{k-1} \text{ or } 2^{k-1}(3^k-1) \pmod{6^k}\},$$

then $a+a' \notin Q_k$ for all $a, a' \in A$. Nathanson [27] has given an upper bound for the size of any set $A \subseteq \{1,2,\ldots,n\}$ with the property that $a+a' \notin Q_k$ for all $a, a' \in A$.

THEOREM 28. Let $k \geq 2$ and $\delta > 0$. For n sufficiently large, if $A \subseteq \{1,2,\ldots,n\}$ satisfies the condition that $a+a' \notin Q_k$ for all $a, a' \in A$ with $a \neq a'$, then either

(1) $A \subseteq \{a \equiv 0 \pmod{2^k}\}$, or

(2) $A \subseteq \{a \equiv 2^{k-1} \pmod{2^k}\}$, or

(3) $|A| < n(1 - (2^k/((2^k-1)\zeta(k))) + \delta) < n/2^k$.

It follows from this result that if $A \subseteq \{1,2,\ldots,2^k m\}$ and $|A| \geq m+1$, then there exist $a, a' \in A$ with $a \neq a'$ and $a+a' \in Q_k$. Note that Filaseta's theorem is the case $k = 2$ of Theorem 28.

Problem 18. Let A be the largest subset of $\{1,2,\ldots,n\}$ such that $a+a' \notin Q_k$ for all $a, a' \in A$ with $a \neq a'$, and A is not of the form (1) or (2) in Theorem 28. Calculate $\limsup |A|/n$.

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