Cassels Bases

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Summary This paper describes several classical constructions of thin bases of finite order in additive number theory, and, in particular, gives a complete presentation of a beautiful construction of Cassels of a class of polynomially asymptotic bases. Some open problems are also discussed.

Keywords Additive basis · Sumset · Thin basis · Polynomially asymptotic basis · Cassels basis · Raikov-Stöhr basis · Jia-Nathanson basis · Additive number theory

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1 Additive Bases of Finite Order

The fundamental object in additive number theory is the *sumset*. If $h \geq 2$ and $A_1, \ldots, A_h$ are sets of integers, then we define the sumset

$$A_1 + \cdots + A_h = \{a_1 + \cdots + a_h : a_i \in A_i \text{ for } i = 1, \ldots, h\}. \tag{1}$$

If $A_1 = A_2 = \cdots = A_h = A$, then the sumset

$$hA = \underbrace{A + A + \cdots + A}_{h \text{ summands}} \tag{2}$$

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is called the *h-fold sumset of A*. If $0 \in A$, then

$$A \subseteq 2A \subseteq \cdots \subseteq hA \subseteq (h+1)A \subseteq \cdots$$

For example,

$$\{0, 1, 4, 5\} + \{0, 2, 8, 10\} = [0, 15]$$

and

$$\begin{aligned}&\{3, 5, 7, 11\} + \{3, 5, 7, 11, 13, 17, 19\}\\ &\quad = \{6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}.\end{aligned}$$

The set A is called a *basis of order h* for the set B if every element of B can be represented as the sum of exactly h not necessarily distinct elements of A, or, equivalently, if $B \subseteq hA$. The set A is an *asymptotic basis of order h* for B if the sumset hA contains all but finitely many elements of B, that is, if $\mathrm{card}(B \backslash hA) < \infty$. The set A is a basis (respectively, asymptotic basis) of finite order for B if A is a basis (respectively, asymptotic basis) of order h for B for some positive integer h. The set A of non-negative integers is a basis of finite order for the non-negative integers only if $0, 1 \in A$.

Many classical results and conjectures in additive number theory state that some "interesting" or "natural" set of non-negative integers is a basis or asymptotic basis of finite order. For example, the Goldbach conjecture asserts that the set of odd prime numbers is a basis of order 2 for the even integers greater than 4. Lagrange's theorem states the set of squares is a basis of order 4 for the non-negative integers $\mathbf{N}_0$. Wieferich proved that the set of non-negative cubes is a basis of order 9 for $\mathbf{N}_0$, and Linnik proved that the set of non-negative cubes is an asymptotic basis of order 7 for $\mathbf{N}_0$. More generally, for any integer $k \geq 2$, Waring's problem, proved by Hilbert in 1909, states that the set of non-negative k-th powers is a basis of finite order for $\mathbf{N}_0$. Vinogradov proved that the set of odd prime numbers is an asymptotic basis of order 3 for the odd positive integers. Nathanson [11] contains complete proofs of all of these results.

Notation: Let $\mathbf{N}$, $\mathbf{N}_0$, and $\mathbf{Z}$ denote the sets of positive integers, non-negative integers, and integers, respectively. For real numbers x and y, we define the intervals of integers $[x, y] = \{n \in \mathbf{Z} : x \leq n \leq y\}$, $(x, y] = \{n \in \mathbf{Z} : x < n \leq y\}$, and $[x, y) = \{n \in \mathbf{Z} : x \leq n < y\}$. For any sets A and A' of integers and any integer c, we define the *difference set*

$$A - A' = \{a - a' : a \in A \text{ and } a' \in A'\}$$

and the *dilation* by c of the set A

$$c * A = \{ca : a \in A\}.$$

Thus, $2 * \mathbf{N}$ is the set of positive even integers, and $2 * \mathbf{N} - \{0, 1\} = \mathbf{N}$.

Denote the cardinality of the set X by $|X|$.

Let f be a complex-valued function on the domain Ω and let g be a positive function on the domain Ω. Usually, Ω is the set of positive integers or the set of all real numbers $x \geq x_0$. We write $f \ll g$ or $f = O(g)$ if there is a number $c > 0$ such that $|f(x)| \leq cg(x)$ for all $x \in \Omega$. We write $f \gg g$ if there is a number $c > 0$ such that $|f(x)| \geq cg(x)$ for all $x \in \Omega$. We write $f = o(g)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

2 A Lower Bound for Bases of Finite Order

For any set A of integers, the *counting function* of A, denoted $A(x)$, counts the number of positive integers in A not exceeding x, that is,

$$A(x) = \sum_{\substack{a\in A\\ 1\leq a\leq x}} 1 = |A \cap [1, x]| .$$

Theorem 1. *Let $h \geq 2$ and let $A = \{a_k\}_{k=1}^{\infty}$ be a set of non-negative integers with $a_k < a_{k+1}$ for all $k \geq 1$. If A is an asymptotic basis of order h then*

$$A(x) \gg x^{1/h} \tag{3}$$

for all sufficiently large real numbers x and

$$a_k \ll k^h \tag{4}$$

for all positive integers k. If A is a basis of order h, then inequality (3) holds for all real numbers $x \geq 1$.

Proof. If A is an asymptotic basis of order h, then there exists an integer n_0 such that every integer $m \geq n_0$ can be represented as the sum of h elements of A. Let $x \geq x_0$ and let n be the integer part of x. Then $A(x) = A(n)$. There are $n - n_0 + 1$ integers m such that

$$n_0 \leq m \leq n.$$

As the elements of A are non-negative integers, it follows that if

$$m = a'_1 + \cdots + a'_h \qquad \text{with } a'_k \in A \text{ for } k = 1, \ldots, h$$

then

$$0 \leq a'_k \leq m \leq n \qquad \text{for } k = 1, \ldots, h.$$

The set A contains exactly $A(n)$ positive integers not exceeding n, and A might also contain 0, hence A contains at most $A(n) + 1$ non-negative integers not

exceeding n. As the number of ways to choose h elements with repetitions from a set of cardinality $A(n)+1$ is $\binom{A(n)+h}{h}$, it follows that

$$n+1-n_0 \leq \binom{A(n)+h}{h} < \frac{(A(n)+h)^h}{h!}$$

and so

$$A(x) = A(n) > (h!(n+1-n_0))^{1/h} - h \gg n^{1/h} \gg x^{1/h}$$

for all sufficiently large x. We have $A(a_k) = k$ if $a_1 \geq 1$ and $A(a_k) = k-1$ if $a_1 = 0$, hence

$$k \geq A(a_k) \gg a_k^{1/h}$$

or, equivalently,

$$a_k \ll k^h$$

for all sufficiently large integers k, hence for all positive integers k.

If A is a basis of order h then $1 \in A$ and so $A(n)/n > 0$ for all $n \geq 1$. Therefore, $A(x) \gg x^{1/h}$ for all $x \geq 1$. This completes the proof. □

Let A be a set of non-negative integers. By Theorem 1, if A is an asymptotic basis of order h then $A(x) \gg x^{1/h}$. If A is an asymptotic basis of order h such that

$$A(x) \ll x^{1/h}$$

then A is called a *thin asymptotic basis* of order h. If $hA = \mathbf{N}_0$ and $A(x) \ll x^{1/h}$ then A is called a *thin basis* of order h. In the next section we construct examples of thin bases.

3 Raikov-Stöhr Bases

In 1937 Raikov and Stöhr independently published the first examples of thin bases for the natural numbers. Their construction is based on the fact that every non-negative integer can be written uniquely as the sum of pairwise distinct powers of 2. The sets constructed in the following theorem will be called *Raikov-Stöhr bases*.

Theorem 2 (Raikov-Stöhr). *Let $h \geq 2$. For $i = 0, 1, \ldots, h-1$, let $W_i = \{i, h+i, 2h+i, \ldots\}$ denote the set of all non-negative integers that are congruent to i modulo h, and let $\mathcal{F}(W_i)$ be the set of all finite subsets of W_i. Let*

$$A_i = \left\{ \sum_{f \in F} 2^f : F \in \mathcal{F}(W_i) \right\}$$

and

$$A = A_0 \cup A_1 \cup \cdots \cup A_{h-1}.$$

Then A is a thin basis of order h.

Proof. Note that for all $i = 0, 1, \ldots, h-1$ we have $0 \in A_i$ since $\emptyset \in \mathcal{F}(W_i)$ and $\sum_{f \in \emptyset} 2^f = 0$. This implies that

$$A_0 + A_1 + \cdots + A_{h-1} \subseteq h\left(\bigcup_{i=0}^{h-1} A_i\right) = hA$$

Moreover, $A_i \cap A_j = \{0\}$ if $0 \le i < j \le h-1$.

First we show that A is a basis of order h. Every positive integer n is uniquely the sum of distinct powers of two, so we can write

$$n = \sum_{j=0}^{\infty} \varepsilon_j 2^j,$$

where the sequence $\{\varepsilon_j\}_{j=0}^{\infty}$ satisfies $\varepsilon_j \in \{0, 1\}$ for all $j \in \mathbf{N}_0$ and $\varepsilon_j = 0$ for all sufficiently large j. Because

$$\sum_{\substack{j=0 \\ j \equiv i \pmod h}}^{\infty} \varepsilon_j 2^j \in A_i,$$

it follows that

$$\begin{aligned} n &= \sum_{j=0}^{\infty} \varepsilon_j 2^j \\ &= \sum_{i=0}^{h-1} \left(\sum_{\substack{j=0 \\ j \equiv i \pmod h}}^{\infty} \varepsilon_j 2^j \right) \\ &\in A_0 + A_1 + \cdots + A_{h-1} \\ &\subseteq hA \end{aligned}$$

and so A is a basis of order h.

We shall compute the counting functions of the sets A_i and A. Let $x \ge 2^{h-1}$. For every $i \in \{0, 1, \ldots, h-1\}$, there is a unique positive integer r such that

$$2^{(r-1)h+i} \le x < 2^{rh+i}.$$

If $a_i \in A_i$ and $a_i \le x$ then there is a set

$$F \subseteq \{i, h+i, \ldots, (r-1)h+i\}$$

such that

$$a_i = \sum_{f \in F} 2^f.$$

The number of such sets F is exactly 2^r. Because $0 \in A_i$, we have

$$A_i(x) \leq 2^r - 1 < 2^r \leq 2^{1-i/h} x^{1/h}$$

and so

$$\begin{aligned} A(x) &= A_0(x) + A_1(x) + \cdots + A_{h-1}(x) \\ &< \left(\sum_{i=0}^{h-1} 2^{1-i/h} \right) x^{1/h} \\ &= \left(\frac{1}{1 - 2^{-1/h}} \right) x^{1/h}. \end{aligned}$$

Thus, A is a thin basis of order h. This completes the proof. □

For $h = 2$, the Raikov-Stöhr construction produces the thin basis $A = A_0 \cup A_1$ of order 2, where

$$A_0 = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, 81, 84, 85, 256, \ldots\}$$

is the set of all finite sums of even powers of 2, and

$$A_1 = \{0, 2, 8, 10, 32, 34, 40, 42, 128, 130, 136, 138, 160, 162, 168, 170, 512, \ldots\}$$

is the set of all finite sums of odd powers of 2.

4 Construction of Thin g-adic Bases of Order h

Lemma 1. *Let $g \geq 2$. Let W be a nonempty set of non-negative integers such that*

$$W(x) = \theta x + O(1)$$

for some $\theta \geq 0$ and all $x \geq 1$. Let $\mathcal{F}(W)$ be the set of all finite subsets of W. Let $A(W)$ be the set consisting of all integers of the form

$$a = \sum_{w \in F} e_w g^w, \tag{5}$$

where $F \in \mathcal{F}(W)$ *and* $e_w \in \{0, 1, \ldots, g-1\}$ *for all* $w \in F$. *Then*

$$x^\theta \ll A(W)(x) \ll x^\theta$$

for all sufficiently large x.

Proof. The nonempty set W is finite if and only if $\theta = 0$, and in this case $A(W)$ is also nonempty and finite, or, equivalently, $1 \ll A(W)(x) \ll 1$.

Suppose that $\theta > 0$ and the set W is infinite. Let $W = \{w_i\}_{i=1}^\infty$, where $0 \le w_1 < w_2 < w_3 < \cdots$. Let $\delta = 0$ if $w_1 \ge 1$ and $\delta = 1$ if $w_1 = 0$. For $x \ge g^{w_1}$, we choose the positive integer k so that

$$g^{w_k} \le x < g^{w_{k+1}}.$$

Then

$$w_k \le \frac{\log x}{\log g} < w_{k+1}$$

and

$$k = W\left(\frac{\log x}{\log g}\right) + \delta = \frac{\theta \log x}{\log g} + O(1),$$

where $W(x)$ is the counting function of the set W.

If $a \in A(W)$ and $a \le x$, then every power of g that appears with a nonzero coefficient in the g-adic representation (5) of a does not exceed g^{w_k}, and so a can be written in the form:

$$a = \sum_{i=1}^{k} e_{w_i} g^{w_i}, \qquad \text{where } e_{w_i} \in \{0, 1, \ldots, g-1\}.$$

There are exactly g^k integers of this form, and so

$$A(W)(x) \le g^k = g^{\frac{\theta \log x}{\log g} + O(1)} \ll x^\theta.$$

Similarly, if a is one of the $g^{k-1} - 1$ positive integers that can be represented in the form:

$$a = \sum_{i=0}^{k-1} e_{w_i} g^{w_i}$$

then

$$a \le \sum_{i=0}^{k-1} (g-1) g^{w_i} \le \sum_{j=0}^{w_{k-1}} (g-1) g^j < g^{w_{k-1}+1} \le g^{w_k} \le x$$

and so

$$A(W)(x) \geq g^{k-1} - 1 \gg x^{\theta}.$$

This completes the proof. □

Theorem 3 (Jia-Nathanson). *Let $g \geq 2$ and $h \geq 2$. Let $W_0, W_1, \ldots, W_{h-1}$ be nonempty sets of non-negative integers such that*

$$\mathbf{N}_0 = W_0 \cup W_1 \cup \cdots \cup W_{h-1}$$

and

$$W_i(x) = \theta_i x + O(1),$$

where $0 \leq \theta_i \leq 1$ for $i = 0, 1, \ldots, h-1$. Let

$$\theta = \max(\theta_0, \theta_1, \ldots, \theta_{h-1}).$$

Let $A(W_0), A(W_1), \ldots, A(W_{h-1})$ be the sets of non-negative integers constructed in Lemma 1. The set

$$A = A(W_0) \cup A(W_1) \cup \cdots \cup A(W_{h-1})$$

is a basis of order h, and

$$A(x) = O\left(x^{\theta}\right).$$

In particular, if

$$W_i(x) = \frac{x}{h} + O(1)$$

for $i = 0, 1, \ldots, h-1$ then $A = A(W_0) \cup A(W_1) \cup \cdots \cup A(W_{h-1})$ is a thin basis of order h.

Note that it is not necessary to assume that the sets $W_0, W_1, \ldots, W_{h-1}$ are pairwise disjoint.

Proof. Every non-negative integer n has a g-adic representation of the form

$$n = \sum_{w=0}^{t} e_w g^w,$$

where $t \geq 0$ and $e_w \in \{0, 1, \ldots, g-1\}$ for $w = 0, 1, \ldots, t$. We define the sets

$$\begin{aligned}
F_0 &= \{w \in \{0, 1, \ldots, t\} : w \in W_0\} \\
F_1 &= \{w \in \{0, 1, \ldots, t\} : w \in W_1 \setminus W_0\} \\
F_2 &= \{w \in \{0, 1, \ldots, t\} : w \in W_2 \setminus (W_0 \cup W_1)\} \\
&\vdots \\
F_{h-1} &= \{w \in \{0, 1, \ldots, t\} : w \in W_{h-1} \setminus (W_0 \cup \cdots \cup W_{h-2})\}.
\end{aligned}$$

Then $F_i \in \mathcal{F}(W_i)$ for all $i = 0, 1, \ldots, h-1$. Since $0 \in A(W_i)$ for $i = 0, 1, \ldots, h-1$, we have

$$n = \sum_{w=0}^{t} e_w g^w = \sum_{i=0}^{h-1} \sum_{w \in F_i} e_w g^w \in A(W_0) + \cdots + A(W_{h-1}) \in hA.$$

Thus, A is a basis of order h.

By Lemma 1,

$$A(W_i)(x) = O\left(x^{\theta_i}\right) = O\left(x^{\theta}\right)$$

for all $i = 0, 1, \ldots, h-1$, and so

$$A(W)(x) \leq \sum_{i=0}^{h-1} A(W_i)(x) = O\left(x^{\theta}\right).$$

If $\theta_i = 1/h$ for all i then $\theta = 1/h$ and A is a thin basis. This completes the proof. □

Consider the case when $W_i = \{w \in \mathbf{N}_0 : w \equiv i \pmod{h}\}$ for $i = 0, 1, \ldots, h-1$. We shall compute an upper bound for the counting functions $A_i(x)$ and $A(x)$. For each i and $x \geq g^i$, choose the positive integer r such that

$$g^{(r-1)h+i} \leq x < g^{rh+i}.$$

Then

$$A_i(x) \leq g^r - 1 < g^r \leq g^{1-(i/h)} x^{1/h}$$

and so

$$A(x) = \sum_{i=0}^{h-1} A_i(x) < \sum_{i=0}^{h-1} g^{1-(i/h)} x^{1/h} = \frac{g-1}{1-g^{-1/h}} x^{1/h}.$$

Applying the mean value theorem to the function $f(x) = x^{1/h}$, we obtain $A(x) < ghx^{1/h}$. In particular, if $g = 2$, we obtain $A(x) < \frac{1}{1-2^{-1/h}} x^{1/h} < 2hx^{1/h}$. This special case is the Raikov-Stöhr construction. For $h = 2$ the Raikov-Stöhr basis $A = \{a_k\}_{k=1}^{\infty}$ with $a_k < a_{k+1}$ for $k \geq 1$ satisfies

$$\frac{A(x)}{\sqrt{x}} < 2 + \sqrt{2} = 3.4142\ldots.$$

Letting $x = a_k$, we obtain

$$\frac{a_k}{k^2} > \frac{3 - 2\sqrt{2}}{2} = 0.0857\ldots.$$

If A is a basis of order h then the order of magnitude of the counting function $A(x)$ must be at least $x^{1/h}$, and there exist thin bases, such as the Raikov-Stöhr

bases and the Jia-Nathanson bases, with exactly this order of magnitude. Two natural constants associated with thin bases of order h are

$$\alpha_h = \inf_{\substack{A \subseteq \mathbf{N}_0 \\ hA = \mathbf{N}_0}} \liminf_{x \to \infty} \frac{A(x)}{x^{1/h}}$$

and

$$\beta_h = \inf_{\substack{A \subseteq \mathbf{N}_0 \\ hA = \mathbf{N}_0}} \limsup_{x \to \infty} \frac{A(x)}{x^{1/h}}$$

Stöhr [16] proved the following lower bound for β_h.

Theorem 4 (Stöhr).

$$\beta_h \geq \frac{\sqrt[h]{h!}}{\Gamma(1 + 1/h)},$$

where $\Gamma(x)$ is the Gamma function.

In particular, $\limsup_{x \to \infty} A(x)/\sqrt{x} \geq \sqrt{8/\pi}$ for every basis A of order 2.

Open Problem 1 *Compute the numbers α_h and β_h for all $h \geq 2$.*

This is an old unsolved problem in additive number theory. Even the numbers α_2 and β_2 are unknown.

5 Asymptotically Polynomial Bases

Let $h \geq 2$, and let $A = \{a_k\}_{k=1}^{\infty}$ be a set of non-negative integers with $a_1 = 0$ and $a_k < a_{k+1}$ for all $k \geq 1$. If A is a basis of order h, then there is a real number λ_2 such that $a_k \leq \lambda_2 k^h$ for all k (Theorem 1). The basis A is called *thin* if there is also a number $\lambda_1 > 0$ such that $a_k \geq \lambda_1 k^h$ for all k. Thus, if A is a thin basis of order h, then there exist positive real numbers λ_1 and λ_2 such that

$$\lambda_1 \leq \frac{a_k}{k^h} \leq \lambda_2$$

for all k. In Theorems 2 and 3 we constructed examples of thin bases of order h for all $h \geq 2$.

The sequence $A = \{a_k\}_{k=0}^{\infty}$ is called *asymptotically polynomial of degree d* if there is a real number $\lambda > 0$ such that $a_k \sim \lambda k^d$ as $k \to \infty$. If A is a basis of order h and if A is also asymptotically polynomial of degree d, then $d \leq h$. We shall describe a beautiful construction of Cassels of a family of additive bases of order h that are asymptotically polynomial of degree h. The key to the construction is the following result, which allows us to embed a sequence of non-negative integers with regular growth into a sequence of non-negative integers with asymptotically polynomial growth.

Theorem 5. *Let $h \geq 2$ and let $A = \{a_k\}_{k=1}^{\infty}$ be a sequence of non-negative integers such that*

$$\liminf_{k\to\infty} \frac{a_{k+1} - a_k}{a_k^{(h-1)/h}} = \alpha > 0.$$

For every real number γ with $0 < \gamma < \alpha$, there exists a sequence $C = \{c_k\}_{k=0}^{\infty}$ of non-negative integers such that C is a supersequence of A and

$$c_k = \left(\frac{\gamma k}{h}\right)^h + O\left(k^{h-1}\right).$$

Proof. Let $B = \{b_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of non-negative integers such that

$$b_k = \left(\frac{\gamma k}{h}\right)^h + O(k^{h-2}).$$

Because $h \geq 2$ and $b_k = (\gamma k/h)^h \left(1 + O\left(k^{-2}\right)\right)$, we have

$$\begin{aligned}
\frac{b_{k+1} - b_k}{b_k^{(h-1)/h}} &= \frac{\left(\frac{\gamma}{h}\right)^h \left((k+1)^h - k^h + O(k^{h-2})\right)}{\left(\frac{\gamma k}{h}\right)^{h-1} \left(1 + O\left(k^{-2}\right)\right)^{(h-1)/h}} \\
&= \frac{\gamma\left(hk^{h-1} + O\left(k^{h-2}\right)\right)}{hk^{h-1}\left(1 + O\left(k^{-2}\right)\right)^{(h-1)/h}} \\
&= \frac{\gamma\left(1 + O\left(k^{-1}\right)\right)}{\left(1 + O\left(k^{-2}\right)\right)^{(h-1)/h}} \\
&= \gamma(1 + o(1))
\end{aligned}$$

and so

$$\lim_{k\to\infty} \frac{b_{k+1} - b_k}{b_k^{(h-1)/h}} = \gamma.$$

Suppose there exist infinitely many k such that, for some integer $m = m(k)$,

$$b_k < a_m < a_{m+1} \leq b_{k+1}.$$

The inequality

$$\frac{b_{k+1} - b_k}{b_k^{(h-1)/h}} > \frac{a_{m+1} - a_m}{b_k^{(h-1)/h}} > \frac{a_{m+1} - a_m}{a_m^{(h-1)/h}}$$

implies that

$$\gamma = \lim_{k\to\infty} \frac{b_{k+1} - b_k}{b_k^{(h-1)/h}} \geq \liminf_{m\to\infty} \frac{a_{m+1} - a_m}{a_m^{(h-1)/h}} \geq \alpha > \gamma$$

which is impossible. Therefore, there exists an integer K such that, for every integer $k \geq K$, the interval $(b_k, b_{k+1}]$ contains at most one element of A.

Choose the integer L such that

$$a_L \leq b_K < a_{L+1}.$$

We define the sequence $C = \{c_k\}_{k=0}^{\infty}$ as follows: Let $c_k = a_k$ for $k = 1, 2, \ldots, L$. For $i \geq 1$, we choose $c_{L+i} \in (b_{K+i-1}, b_{K+i}]$ as follows: If the interval $(b_{K+i-1}, b_{K+i}]$ contains the element a_ℓ from the sequence A then $c_{L+i} = a_\ell$. Otherwise, let $c_{L+i} = b_{K+i}$. As the interval $(b_{K+i-1}, b_{K+i}]$ contains at most one element of A for all $i \geq 1$, and since every element a_k of A with $k > L$ is contained in some interval of the form $(b_{K+i-1}, b_{K+i}]$ with $i \geq 1$, it follows that A is a subsequence of C. Moreover, for every $k \geq L + 1$,

$$b_{k-L+K-1} < c_k \leq b_{k-L+K}.$$

As

$$b_{k-L+K} = \left(\frac{\gamma}{h}\right)^h (k - L + K)^h + O(k^{h-2}) = \left(\frac{\gamma k}{h}\right)^h + O\left(k^{h-1}\right)$$

and, similarly, $b_{k-L+K-1} = (\gamma k/h)^h + O\left(k^{h-1}\right)$, it follows that

$$c_k = \left(\frac{\gamma k}{h}\right)^h + O\left(k^{h-1}\right).$$

This completes the proof. □

6 Bases of Order 2

In this section we describe Cassels' construction in the case $h = 2$. We need the following convergence result.

Lemma 2. *Let* $0 < \alpha < 1$. *If* $\{q_k\}_{k=1}^{\infty}$ *is a sequence of positive integers such that*

$$\lim_{k\to\infty} \frac{q_{k-1}}{q_k} = \alpha$$

then

$$\lim_{k\to\infty} \frac{q_1 + q_2 + \cdots + q_k}{q_k} = \frac{1}{1-\alpha}.$$

Proof. For every non-negative integer j we have

$$\lim_{k\to\infty} \frac{q_{k-j}}{q_k} = \lim_{k\to\infty} \prod_{i=0}^{j-1} \frac{q_{k-i-1}}{q_{k-i}} = \alpha^j. \tag{6}$$

Let β be a real number such that $\alpha < \beta < 1$. For every $\varepsilon > 0$ there exists a number $K = K(\beta, \varepsilon)$ such that

$$\frac{q_{k-1}}{q_k} < \beta \qquad \text{for all } k \geq K \tag{7}$$

and

$$\beta^K < \frac{(1-\beta)\varepsilon}{4}. \tag{8}$$

If $k \geq K$ and $k - K = r$ then

$$q_k > \beta^{-1} q_{k-1} > \beta^{-2} q_{k-2} > \cdots > \beta^{-r} q_{k-r} = \beta^{K-k} q_K = c\beta^{-k},$$

where $c = \beta^K q_K > 0$, and so

$$\lim_{k\to\infty} q_k = \infty. \tag{9}$$

If $0 \leq j \leq k - K + 1$, then inequality (7) implies

$$\frac{q_{k-j}}{q_k} = \prod_{i=0}^{j-1} \frac{q_{k-i-1}}{q_{k-i}} < \beta^j.$$

For $k \geq 2K$ we obtain

$$\begin{aligned}
\left| \frac{q_1 + q_2 + \cdots + q_k}{q_k} - \frac{1}{1-\alpha} \right| &= \left| \sum_{j=0}^{k-1} \frac{q_{k-j}}{q_k} - \sum_{j=0}^{\infty} \alpha^j \right| \\
&\leq \sum_{j=0}^{K-1} \left| \frac{q_{k-j}}{q_k} - \alpha^j \right| + \sum_{j=K}^{k-K+1} \frac{q_{k-j}}{q_k} + \sum_{j=k-K+2}^{k-1} \frac{q_{k-j}}{q_k} + \sum_{j=K}^{\infty} \alpha^j \\
&< \sum_{j=0}^{K-1} \left| \frac{q_{k-j}}{q_k} - \alpha^j \right| + \sum_{j=K}^{k-K+1} \beta^j + \sum_{j=1}^{K-2} \frac{q_j}{q_k} + \sum_{j=K}^{\infty} \beta^j \\
&< \sum_{j=0}^{K-1} \left| \frac{q_{k-j}}{q_k} - \alpha^j \right| + \sum_{j=1}^{K-2} \frac{q_j}{q_k} + \frac{2\beta^K}{1-\beta}
\end{aligned}$$

It follows from (6), (9), and (8) that for $j = 0, 1, \ldots, K-1$ and all sufficiently large k

$$\left| \frac{q_{k-j}}{q_k} - \alpha^j \right| < \frac{\varepsilon}{4K}$$

and

$$\frac{q_j}{q_k} < \frac{\varepsilon}{4K}$$

and so

$$\left| \frac{q_1 + q_2 + \cdots + q_k}{q_k} - \frac{1}{1-\alpha} \right| < \varepsilon.$$

This completes the proof. □

Theorem 6. *Let $\{q_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ be sequences of positive integers such that*

$$q_1 = 1 \tag{10}$$

and, for all $i \geq 2$,

$$(q_{i-1}, q_i) = (q_{i-1}, q_{i+1}) = 1 \tag{11}$$

$$m_{i-1} \geq q_i + q_{i+1} - 2 \tag{12}$$

and

$$m_{i+1}q_{i+1} \geq m_i q_i + m_{i-1}q_{i-1}. \tag{13}$$

Define the sequences $\{Q_k\}_{k=1}^{\infty}$ of non-negative integers and $\{A_k\}_{k=1}^{\infty}$ of finite arithmetic progressions of non-negative integers by:

$$Q_k = \sum_{i=1}^{k-1} m_i q_i$$

and

$$A_k = Q_k + q_k * [0, m_k].$$

Let

$$A = \bigcup_{k=1}^{\infty} A_k = \{a_n\}_{n=0}^{\infty}$$

where $a_0 = 0 < a_1 < a_2 < \cdots$. Then A is a basis of order 2, and, for every positive integer K, the set $\bigcup_{k=K}^{\infty} A_k$ is an asymptotic basis of order 2.

Let $A(x)$ be the counting function of the set A, and let $M_k = \sum_{i=1}^{k-1} m_i$ for $k \geq 1$. If $M_k \leq n \leq M_{k+1}$, then

$$a_n = Q_k + (n - M_k)q_k. \tag{14}$$

If $Q_k \le x \le Q_{k+1}$, *then*

$$A(x) = M_k + \left[\frac{x - Q_k}{q_k}\right]. \tag{15}$$

Proof. As $Q_{k+1} - Q_k = m_k q_k$, it follows that

$$\{Q_k, Q_{k+1}\} \subseteq A_k \subseteq [Q_k, Q_{k+1}]$$

and

$$A_k = Q_{k+1} - q_k * [0, m_k].$$

Also, $Q_1 = 0$, $Q_2 = m_1 q_1 = m_1$, and $A_1 = [0, m_1]$, hence

$$[2Q_1, 2Q_2] = [0, 2m_1] = 2A_1.$$

We shall prove that

$$[2Q_k, 2Q_{k+1}] \subseteq A_{k-1} + (A_k \cup A_{k+1}) \subseteq 2\,(A_{k-1} \cup A_k \cup A_{k+1}) \tag{16}$$

for all $k \ge 2$.

Let $n \in [2Q_k, 2Q_{k+1}]$. There are two cases. In the first case we have

$$2Q_k \le n \le Q_k + Q_{k+1} - (q_k - 1)q_{k-1}. \tag{17}$$

Because $(q_k, q_{k-1}) = 1$, there is a unique integer r such that

$$n \equiv 2Q_k - rq_{k-1} \pmod{q_k}$$

and, by (12),

$$0 \le r \le q_k - 1 \le m_{k-1}. \tag{18}$$

Then $Q_k - rq_{k-1} \in A_{k-1}$. There is a unique integer s such that

$$sq_k = n - 2Q_k + rq_{k-1}.$$

It follows from (17) and (18) that

$$0 \le n - 2Q_k + rq_{k-1} \le Q_{k+1} - Q_k = m_k q_k,$$

and so

$$0 \le s \le m_k.$$

Therefore, $Q_k + sq_k \in A_k$ and

$$n = (Q_k - rq_{k-1}) + (Q_k + sq_k) \in A_{k-1} + A_k.$$

In the second case we have

$$Q_k + Q_{k+1} - (q_k - 1)q_{k-1} + 1 \leq n \leq 2Q_{k+1}. \tag{19}$$

The set $R = [q_k - 1, q_k + q_{k+1} - 2]$ is a complete set of representatives of the congruence classes modulo q_{k+1}. Because $(q_{k-1}, q_{k+1}) = 1$, it follows that there is a unique integer $r \in R$ such that

$$n \equiv Q_k + Q_{k+1} - rq_{k-1} \pmod{q_{k+1}}.$$

Inequality (12) implies that

$$0 \leq q_k - 1 \leq r \leq q_k + q_{k+1} - 2 \leq m_{k-1} \tag{20}$$

and so $Q_k - rq_{k-1} \in A_{k-1}$. There is a unique integer t such that

$$tq_{k+1} = n - Q_k - Q_{k+1} + rq_{k-1},$$

Inequalities (19), (20), and (13) imply that

$$tq_{k+1} \geq (r - q_k + 1)q_{k-1} + 1 \geq 1$$

and

$$tq_{k+1} \leq Q_{k+1} - Q_k + rq_{k-1} \leq m_k q_k + m_{k-1}q_{k-1} \leq m_{k+1}q_{k+1},$$

and so

$$1 \leq t \leq m_{k+1}.$$

Therefore, $Q_{k+1} + tq_{k+1} \in A_{k+1}$ and

$$n = (Q_k - rq_{k-1}) + (Q_{k+1} + tq_{k+1}) \in A_{k-1} + A_{k+1}.$$

This proves (16). It follows that $\bigcup_{k=1}^{\infty} A_k$ is a basis of order 2. Moreover, for every positive integer K,

$$[2Q_{K+1}, \infty) \subseteq 2\left(\bigcup_{k=K}^{\infty} A_k\right)$$

and so $\bigcup_{k=K}^{\infty} A_k$ is an asymptotic basis of order 2.

Let $A = \{a_n\}_{n=0}^{\infty}$, where $a_0 = 0 < a_1 < a_2 < \cdots$, and let $A(x)$ be the counting function of the set A. Formulas (14) and (15) are immediate consequences of the construction of the set A. This completes the proof. □

Theorem 7. *Let* $0 < \alpha < 1$ *and let* $\{q_i\}_{i=1}^{\infty}$ *be a sequence of positive integers with* $q_1 = 1$ *such that, for all* $i \geq 2$,

$$(q_{i-1}, q_i) = (q_{i-1}, q_{i+1}) = 1 \tag{21}$$

$$q_{i+1}(q_{i+2} + q_{i+3}) \geq q_i(q_{i+1} + q_{i+2}) + q_{i-1}(q_i + q_{i+1}) \tag{22}$$

and

$$\lim_{i\to\infty} \frac{q_{i-1}}{q_i} = \alpha. \tag{23}$$

Define the sequences $\{Q_k\}_{k=1}^{\infty}$ *of non-negative integers and* $\{A_k\}_{k=1}^{\infty}$ *of finite arithmetic progressions of non-negative integers by:*

$$Q_k = \sum_{i=1}^{k-1} q_i(q_{i+1} + q_{i+2})$$

and

$$A_k = Q_k + q_k * [0, q_{k+1} + q_{k+2}].$$

Let

$$A = \bigcup_{k=1}^{\infty} A_k = \{a_n\}_{n=0}^{\infty},$$

where $a_0 = 0 < a_1 < a_2 < \cdots$. *Then* A *is a basis of order 2 such that*

$$\liminf_{k\to\infty} \frac{a_{n+1} - a_n}{n} \geq \frac{\alpha^2(1-\alpha)}{1+\alpha} > 0.$$

Note that the sequence $\{q_i\}_{i=1}^{\infty}$ of Fibonacci numbers defined by $q_1 = q_2 = 1$ and $q_{i+2} = q_{i+1} + q_i$ for $i \geq 1$ satisfies the conditions of Theorem 7 with $\alpha = (\sqrt{5} - 1)/2$.

Proof. For every integer $i \geq 1$ we define the positive integer $m_i = q_{i+1} + q_{i+2}$. Inequality (22) implies that the sequence $\{m_i\}_{i=1}^{\infty}$ satisfies the hypotheses of Theorem 6, and so A is a basis of order 2. For $k \geq 1$ we define:

$$M_k = \sum_{i=1}^{k-1} m_i = \sum_{i=1}^{k-1} (q_{i+1} + q_{i+2}).$$

Then $\{M_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers. For every positive integer n there is a unique integer k such that

$$M_k \leq n < M_{k+1}.$$

By (14) we have

$$a_n = Q_k + (n - M_k)q_k$$

and so

$$a_{n+1} - a_n = q_k,$$

hence

$$\frac{a_{n+1} - a_n}{n} = \frac{q_k}{n} > \frac{q_k}{M_{k+1}}.$$

Condition (23) implies that $\lim_{k\to\infty} q_k = \infty$. As

$$\begin{aligned}\frac{M_{k+1}}{q_k} &= \sum_{i=1}^{k} \frac{q_{i+1} + q_{i+2}}{q_k} \\ &= \sum_{i=2}^{k+1} \frac{q_i}{q_k} + \sum_{i=3}^{k+2} \frac{q_i}{q_k} \\ &= 2\sum_{i=1}^{k} \frac{q_i}{q_k} + 2\frac{q_{k+1}}{q_k} + \frac{q_{k+2}}{q_k} - 2\frac{q_1}{q_k} - \frac{q_2}{q_k},\end{aligned}$$

it follows from Lemma 2 that

$$\lim_{k\to\infty} \frac{M_{k+1}}{q_k} = \frac{2}{1-\alpha} + \frac{2}{\alpha} + \frac{1}{\alpha^2} = \frac{1+\alpha}{\alpha^2(1-\alpha)}.$$

Therefore,

$$\liminf_{k\to\infty} \frac{a_{n+1} - a_n}{n} \geq \lim_{k\to\infty} \frac{q_k}{M_{k+1}} = \frac{\alpha^2(1-\alpha)}{1+\alpha} > 0.$$

This completes the proof. □

Theorem 8 (Cassels). *There exist a basis $C = \{c_n\}_{n=0}^{\infty}$ of order 2 and a real number $\lambda > 0$ such that $c_n = \lambda n^2 + O(n)$.*

Proof. By Theorem 7, there exists a basis $A = \{a_n\}_{n=0}^{\infty}$ of order 2 such that $\liminf_{n\to\infty}(a_{n+1} - a_n)/n > 0$. Applying Theorem 1 with $h = 2$, we see that $a_n \ll n^2$ and so $\liminf_{n\to\infty}(a_{n+1} - a_n)/a_n^{1/2} > 0$. Applying Theorem 5 with $h = 2$, we obtain a sequence $C = \{c_n\}_{n=0}^{\infty}$ of non-negative integers and a positive real number λ such that C is a supersequence of A and $c_n = \gamma n^2 + O(n)$. This completes the proof. □

7 Bases of Order $h \geq 3$

We start with Cassels' construction of a finite set C of integers such that the elements of C are widely spaced and C is a basis of order h for a long interval of integers. The construction uses a perturbation of the g-adic representation.

Lemma 3. *Let $h \geq 3$. Let v and L be positive integers with $L \geq h$. Define*

$$g = 2^{h+1}v.$$

Let $C = C(v, L)$ denote the finite set consisting of the following integers:

$$\begin{array}{ll}
g^h + eg^{h-1} + 2vg^{h-2} + e & \textit{for } 0 \leq e < g, \\
(i+1)g^h + eg^{h-1} + eg^i & \textit{for } 0 \leq i \leq h-3 \textit{ and } 0 \leq e < g, \\
(h-1)g^h + (4vq + r)g^{h-1} + (4vq + r)g^{h-2} & \textit{for } 0 \leq q < 2^{h-1} \textit{ and } 0 \leq r < 2v, \\
hg^h + \ell g^{h-1} & \textit{for } 0 \leq \ell < Lg.
\end{array}$$

Then

(i) The h-fold sumset hC contains every integer n in the interval

$$\left[\left(\frac{h^2+3h-2}{2}\right)g^h, \left(\frac{h(h+1)}{2} + L\right)g^h\right).$$

(ii) If $c \in C$ then

$$g^h \leq c < (h+L)g^h.$$

If $c \geq hg^h$, then $c \equiv 0 \pmod{g^{h-1}}$.

(iii) If $c, c' \in C$ and $c \neq c'$, then

$$|c - c'| \geq vg^{h-2} - g.$$

(iv) If $c \in C$ and y is any integer such that

$$y \equiv -vg^{h-2} \pmod{4vg^{h-2}}$$

then

$$|c - y| \geq vg^{h-2} - g.$$

Proof. (i) Every non-negative integer n has a unique g-adic representation in the form:

$$n = e_{h-1}g^{h-1} + e_{h-2}g^{h-2} + \cdots + e_1 g + e_0, \tag{24}$$

where $e_{h-1} \geq 0$ and

$$0 \leq e_j < g \qquad \text{for } j = 0, 1, \ldots, h-2.$$

If n satisfies the inequality

$$\left(\frac{h^2+3h-2}{2}\right)g^h \le n < \left(\frac{h(h+1)}{2}+L\right)g^h$$

then e_{h-1} satisfies the inequality

$$\left(\frac{h^2+3h-2}{2}\right)g \le e_{h-1} < \left(\frac{h(h+1)}{2}+L\right)g. \tag{25}$$

The digit e_{h-2} satisfies the inequality $0 \le e_{h-2} < g = 4v2^{h-1}$. There are two cases, which depend on the remainder of e_{h-2} when divided by $4v$.

In the first case, we have

$$e_{h-2} = 4vq + r \qquad \text{with } 0 \le q < 2^{h-1} \text{ and } 0 \le r < 2v.$$

Rearranging the g-adic representation (24), we obtain

$$\begin{aligned} n = {} & \left((h-1)g^h + (4vq+r)g^{h-1} + (4vq+r)g^{h-2}\right) \\ & + \sum_{i=0}^{h-3}\left((i+1)g^h + e_i g^{h-1} + e_i g^i\right) + \left(hg^h + \ell g^{h-1}\right), \end{aligned} \tag{26}$$

where

$$\ell = e_{h-1} - \sum_{i=0}^{h-2} e_i - \frac{h(h+1)g}{2}.$$

Inequality (25) implies that

$$\ell \ge \left(\frac{h^2+3h-2}{2}\right)g - (h-1)(g-1) - \frac{h(h+1)g}{2} = h-1 > 0$$

and

$$\ell < \left(\frac{h(h+1)}{2}+L\right)g - \frac{h(h+1)g}{2} = Lg$$

and so $hg^h + \ell g^{h-1} \in C$. Thus, (26) is a representation of n as the sum of h elements of C, that is, $n \in hC$.

In the second case, we have

$$e_{h-2} = 4vq + r + 2v \qquad \text{with } 0 \le q < 2^{h-1} \text{ and } 0 \le r < 2v.$$

From the g-adic representation (24), we obtain

$$\begin{aligned} n = &\Big((h-1)g^h + (4vq+r)g^{h-1} + (4vq+r)g^{h-2}\Big) \\ &+ \sum_{i=1}^{h-3}\Big((i+1)g^h + e_i g^{h-1} + e_i g^i\Big) \\ &+ \Big(g^h + e_0 g^{h-1} + 2vg^{h-2} + e_0\Big) + \Big(hg^h + \ell g^{h-1}\Big), \end{aligned} \tag{27}$$

where

$$\ell = e_{h-1} - (e_{h-2} - 2v) - \sum_{i=0}^{h-3} e_i - \left(\frac{h(h+1)}{2}\right) g.$$

As in the first case, inequality (25) implies that $0 < h-1 \le \ell < Lg$ and so $hg^h + \ell g^{h-1} \in C$. Thus, (27) is a representation of n as the sum of h elements of C, that is, $n \in hC$. This proves (i).

To prove (ii), we observe that the smallest element of C is g^h and the largest is $hg^h + (Lg-1)g^{h-1} < (h+L)g^h$. If $c \in C$ and $c \ge hg^h$ then $c = hg^h + \ell g^{h-1}$ for some non-negative integer $\ell < Lg$, hence $c \equiv 0 \pmod{g^{h-1}}$.

To prove (iii), we assert that every integer $c \in C$ satisfies an inequality of the form:

$$4svg^{h-2} \le c < (4s+2)vg^{h-2} + g \tag{28}$$

for some non-negative integer s. There are four cases to check.

If $c = g^h + eg^{h-1} + 2vg^{h-2} + e$ with $0 \le e < g$ then we choose $s = 2^{h-1}(g+e)$. Because

$$4svg^{h-2} = g^h + eg^{h-1}$$

and

$$(4s+2)vg^{h-2} + g = g^h + eg^{h-1} + 2vg^{h-2} + g$$

it follows that c satisfies (28).

If $c = (i+1)g^h + eg^{h-1} + eg^i$ with $0 \le e < g$ and $0 \le i \le h-3$ then c satisfies (28) with $s = 2^{h-1}((i+1)g + e)$.

If $c = (h-1)g^h + (4vq+r)g^{h-1} + (4vq+r)g^{h-2}$ with $0 \le q < 2^{h-1}$ and $0 \le r < 2v$ then c satisfies (28) with $s = 2^{h-1}((h-1)g + 4vq + r) + q$.

If $c = hg^h + \ell g^{h-1}$ with $0 \le \ell < Lg$ then c satisfies (28) with $s = 2^{h-1}(hg + \ell)$.

This proves (28). It follows that the distance between elements of C that satisfy inequality (28) for different values of s is at least $2vg^{h-2} - g$. If c and c' are distinct elements of C that satisfy inequality (28) for the same value of s, and if $c' < c$, then we must have

$$0 < c - c' < 2vg^{h-2} + g.$$

This can happen only if $c = g^h + eg^{h-1} + 2vg^{h-2} + e$ and $c' = g^h + eg^{h-1} + e$ with $0 \le e < g$, and so $c - c' = 2vg^{h-2}$. This proves (iii).

Finally, to prove (iv), we observe that if $y \equiv -vg^{h-2} \pmod{4vg^{h-2}}$ then $y = 4s'vg^{h-2} - vg^{h-2}$ for some integer s', and the distance between y and any integer satisfying an inequality of the form (28) is at least $vg^{h-2} - g$. This completes the proof of the Lemma. □

Lemma 4. *For $h \geq 3$, let $v_i = 2^i$ and $g_i = 2^{h+1}v_i = 2^{i+h+1}$ for $i = 0, 1, 2, \ldots$. Then*

$$p_j = \sum_{i=0}^{j} v_i g_i^{h-2} < g_j^h.$$

Proof. We compute p_j explicitly as follows:

$$\begin{aligned} p_j &= \sum_{i=0}^{j} v_i g_i^{h-2} = \sum_{i=0}^{j} 2^i \left(2^{i+h+1}\right)^{h-2} = 2^{(h-2)(h+1)} \sum_{i=0}^{j} 2^{(h-1)i} \\ &= 2^{(h-2)(h+1)} \left(\frac{2^{(h-1)(j+1)} - 1}{2^{h-1} - 1}\right) = \frac{2^{h^2+hj-j-3} - 2^{h^2-h-2}}{2^{h-1} - 1} \\ &< 2^{h(j+h+1)} = g_j^h \end{aligned}$$

because, for $h \geq 3$,

$$\begin{aligned} 2^{h^2+hj-j-3} + 2^{h^2+hj+h} &< 2^{h^2+hj+h+1} < 2^{h^2+hj+2h-1} \\ &< 2^{h^2+hj+2h-1} + 2^{h^2-h-2}. \end{aligned}$$

□

Theorem 9. *Let $h \geq 3$. There exists a strictly increasing sequence $A = \{a_k\}_{k=1}^{\infty}$ of non-negative integers such that A is a basis of order h and*

$$\liminf_{k\to\infty} \frac{a_{k+1} - a_k}{a_k^{(h-1)/h}} \geq \frac{1}{2^{3h-1}}.$$

Proof. Let

$$A(-1) = \left[0, 2^{h^2+2h}\right].$$

We define

$$L = 2^{2h} - h - 1$$

and, for $i = 0, 1, 2, \ldots$,

$$\begin{aligned} v_i &= 2^i \\ g_i &= 2^{h+1}v_i = 2^{i+h+1} \end{aligned}$$

and

$$p_j = \sum_{i=0}^{j} v_i g_i^{h-2}.$$

For $j = 0, 1, 2, \ldots$, let

$$A(j) = p_j + C(v_j, L),$$

where $C(v_j, L)$ is the finite set of positive integers constructed in Lemma 3. We begin by proving that

$$A = \bigcup_{j=-1}^{\infty} A(j)$$

is a basis of order h.

First, we observe that

$$I(-1) = \left[0, h2^{h^2+2h}\right] = hA(-1) \subseteq hA$$

and, by Lemma 3,

$$I(j) = \left[hp_j + \left(\frac{h^2+3h-2)}{2}\right) g_j^h, hp_j + \left(\frac{h(h+1)}{2} + L\right) g_j^h\right) \subseteq hA(j)$$

for $j = 0, 1, 2, \ldots$. As $h^2 + 3h - 2 \leq 2^{h+1}$ for $h \geq 3$, it follows that

$$\begin{aligned} hp_0 + \left(\frac{h^2+3h-2)}{2}\right) g_0^h &= h2^{(h+1)(h-2)} + \left(\frac{h^2+3h-2}{2}\right) 2^{h(h+1)} \\ &\leq h2^{h^2-h-2} + 2^{h^2+2h} \\ &\leq h2^{h^2+2h} \end{aligned}$$

and so the intervals $I(-1)$ and $I(0)$ overlap. Similarly, for $j \geq 0$ the intervals $I(j)$ and $I(j+1)$ overlap if

$$hp_{j+1} + \left(\frac{h^2+3h-2}{2}\right) g_{j+1}^h \leq hp_j + \left(\frac{h(h+1)}{2} + L\right) g_j^h. \tag{29}$$

Because $v_{j+1} = 2v_j$ and $g_{j+1} = 2g_j$, we have

$$p_{j+1} - p_j = v_{j+1} g_{j+1}^{h-2} = 2^{h+j-1} g_j^{h-2} = \frac{g_j^h}{2^{h+j+3}}.$$

Rearranging inequality (29) and dividing by g_j^h, we see that it suffices to prove that

$$\frac{h}{2^{h+j+3}} + \left(\frac{h^2+3h-2}{2}\right) 2^h \leq \frac{(h-2)(h+1)}{2} + 2^{2h}.$$

This follows immediately from the inequalities $h^2 + 3h - 2 \leq 2^{h+1}$ and

$$\frac{h}{2^{h+j+3}} \leq 2 \leq \frac{(h-2)(h+1)}{2}$$

for $j \geq 0$ and $h \geq 3$. Thus, the set A is a basis of order h.

Next, we show that the elements of A are widely spaced. Let $a, a' \in A$ with $a' \neq a$ and $a \in A(j)$ and $a' \in A(j')$ for $j, j' \geq 0$. We shall prove that

$$|a - a'| \geq v_j g_j^{h-2} - g_j.$$

Suppose not. If $j = j'$ then there exist $c, c' \in C(v_j, L)$ with $c \neq c'$ such that $a = p_j + c$ and $a' = p_j + c'$. By Lemma 3 (iii) we have $|a - a'| = |c - c'| \geq v_j g_j^{h-2} - g_j$. Thus, if $|a - a'| < v_j g_j^{h-2} - g_j$ then $j \neq j'$.

The sequences $\{p_j\}_{j=0}^{\infty}$ and $\{g_j\}_{j=0}^{\infty}$ are strictly increasing sequences of positive integers. If $j < j'$, then $v_j g_j^{h-3} < v_{j'} g_{j'}^{h-3}$ and so

$$v_j g_j^{h-2} - g_j = (v_j g_j^{h-3} - 1) g_j < (v_{j'} g_{j'}^{h-3} - 1) g_{j'} = v_{j'} g_{j'}^{h-2} - g_{j'}.$$

Thus, if $j < j'$ and $|a - a'| < v_j g_j^{h-2} - g_j$ then also $|a - a'| < v_{j'} g_{j'}^{h-2} - g_{j'}$. Therefore, without loss of generality, we can assume that $j' < j$.

By Lemma 3 (ii) we have $a \geq p_j + g_j^h$ and $a' < p_{j'} + (h+L) g_{j'}^h$. The inequality $|a - a'| < v_j g_j^{h-2} - g_j$ implies that

$$\begin{aligned} a' &> a - v_j g_j^{h-2} + g_j > p_j + g_j^h - v_j g_j^{h-2} \\ &= p_{j-1} + g_j^h = p_{j-1} + 2^h g_{j-1}^h > p_{j-1} + h g_{j-1}^h. \end{aligned}$$

Combining the upper bound in Lemma 3 (ii) with Lemma 4, we get

$$a' < p_{j'} + (h + L) g_{j'}^h < (h + 1 + L) g_{j'}^h = 2^{2h} g_{j'}^h = 2^h g_{j'+1}^h = g_{j'+2}^h.$$

As $g_j^h < a' < g_{j'+2}^h$, we see that $j' < j < j' + 2$ and so $j = j' + 1$ and $a' = p_{j-1} + c'$ for some $c' \in C(v_{j-1}, L)$ with $c' \geq h g_{j-1}^h$. By Lemma 3 (ii), we have $c' \equiv 0 \pmod{g_{j-1}^{h-1}}$ and so

$$a' = p_{j-1} + c' \equiv p_{j-1} = p_j - v_j g_j^{h-2} \pmod{g_{j-1}^{h-1}}.$$

Since

$$g_{j-1}^{h-1} = 2^{h+j} g_{j-1}^{h-2} = 4 v_j 2^{h-2} g_{j-1}^{h-2} = 4 v_j g_j^{h-2}$$

it follows that

$$y = a' - p_j \equiv -v_j g_j^{h-2} \pmod{4 v_j g_j^{h-2}}.$$

There exists $c \in C(v_j, L)$ such that $a = p_j + c$. Lemma 3 (iv) implies that

$$|a - a'| = |c - (a' - p_j)| = |c - y| \geq v_j g_j^{h-2} - g_j$$

which is a contradiction. This proves that if $a, a' \in A \setminus A(-1)$ with $a \neq a'$ and $a \in A(j)$, then $|a - a'| \geq v_j g_j^{h-2} - g_j$.

From Lemmas 3 (ii) and 4 we also have

$$a = p_j + c < g_j^h + (h + L)g_j^h = 2^{2h} g_j^h = (4g_j)^h$$

and so $a^{(h-1)/h} < (4g_j)^{h-1}$ and

$$\frac{|a - a'|}{a^{(h-1)/h}} > \frac{v_j g_j^{h-2} - g_j}{(4g_j)^{h-1}} = \frac{v_j}{4^{h-1} g_j} - \frac{1}{4^{h-1} g_j^{h-2}} = \frac{1}{2^{3h-1}} - \frac{1}{4^{h-1} g_j^{h-2}}.$$

Writing A as a strictly increasing sequence $A = \{a_k\}_{k=1}^{\infty}$ of non-negative integers, we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \frac{a_{k+1} - a_k}{a_k^{(h-1)/h}} &\geq \liminf_{\substack{a,a' \in A\setminus A(-1) \\ a \neq a'}} \frac{|a - a'|}{a^{(h-1)/h}} \\
&\geq \liminf_{j\to\infty} \left(\frac{1}{2^{3h-1}} - \frac{1}{4^{h-1} g_j^{h-2}} \right) \\
&= \frac{1}{2^{3h-1}}.
\end{aligned}$$

This completes the proof. □

Theorem 10 (Cassels). *For every integer $h \geq 3$ there exist a basis $C = \{c_n\}_{n=0}^{\infty}$ of order h and real number $\lambda > 0$ such that $c_n = \lambda n^h + O\left(n^{h-1}\right)$.*

Proof. This follows immediately from Theorems 9 and 5. □

Open Problem 2 *Let $h \geq 2$. Does there exist a basis $C = \{c_n\}_{n=0}^{\infty}$ of order h such that $c_n = \gamma n^h + o(n^{h-1})$ for some $\gamma > 0$?*

Open Problem 3 *Let $h \geq 2$. Does there exist a basis $C = \{c_n\}_{n=0}^{\infty}$ of order h such that $c_n = \gamma n^h + O(n^{h-2})$ for some $\gamma > 0$?*

Open Problem 4 *Let $h \geq 2$. Compute or estimate*

$$\sup\{\lambda > 0 : \textit{ there exists a basis } C = \{c_n\}_{n=0}^{\infty} \textit{ of order } h \textit{ such that } c_n \sim \lambda n^h\}.$$

8 Notes

Raikov [13] and Stöhr [15] independently constructed the first examples of thin bases of order h. Another early, almost forgotten construction of thin bases is due to Chartrovsky [3]. The g-adic generalization of the Raikov-Stöhr construction appears in work of Jia and Nathanson [8, 9] on minimal asymptotic bases. The currently "thinest" bases of finite order appear in recent articles by Hofmeister [7] and Blomer [1]. An old but still valuable survey of combinatorial problems in additive number theory is Stöhr [16].

The classical bases in additive number theory are the squares, cubes, and, for every integer $k \geq 4$, the kth powers of non-negative integers, and also the sets of polygonal numbers and of prime numbers. Using probability arguments, one can prove that all of the classical bases contain thin subsets that are bases of order h for sufficiently large h (Choi-Erdős-Nathanson [4], Erdős-Nathanson [5], Nathanson [10], Wirsing[18], and Vu [17]).

The construction in this paper of polynomially asymptotic thin bases of order h appeared in the classic article of Cassels [2] in 1957. There is a recent quantitative improvement by Schmitt [14], and also related work on Cassels bases by Grekos, Haddad, Helou, and Pihko [6] and Nathanson [12].

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