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# Thin bases in additive number theory

Melvyn B. Nathanson *

*Department of Mathematics, Lehman College (CUNY), Bronx, NY 10468, United States*
*CUNY Graduate Center, New York, NY 10016, United States*

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ABSTRACT

The set $\mathcal{A}$ of nonnegative integers is called a basis of order $h$ if every nonnegative integer can be represented as the sum of exactly $h$ not necessarily distinct elements of $\mathcal{A}$. An additive basis $\mathcal{A}$ of order $h$ is called thin if there exists $c > 0$ such that the number of elements of $\mathcal{A}$ not exceeding $x$ is less than $cx^{1/h}$ for all $x \geq 1$. This paper describes a construction of Shatrovskiĭ of thin bases of order $h$.



## 1. Additive bases of finite order

The central theme in additive number theory is the representation of a nonnegative integer as the sum of a bounded number of elements chosen from a fixed set of integers, such as the squares, the cubes, or the primes. Let $\mathbf{N}_0$ and $\mathbf{Z}$ denote the sets of nonnegative integers and integers, respectively. For real numbers $a$ and $b$ with $a \leq b$, let $[a, b]$ denote the interval of integers between $a$ and $b$, that is, $[a, b] = \{n \in \mathbf{Z} : a \leq n \leq b\}$.

Let $h \geq 2$, and let $(\mathcal{A}_1, \ldots, \mathcal{A}_h)$ be an $h$-tuple of subsets of $\mathbf{N}_0$. We define the *sumset*

$$\mathcal{A}_1 + \cdots + \mathcal{A}_h = \{a_1 + \cdots + a_h : a_i \in \mathcal{A}_i \text{ for } i = 1, \ldots, h\}.$$

The *h-fold sumset* of a set $\mathcal{A}$ of nonnegative integers is

$$h\mathcal{A} = \underbrace{\mathcal{A} + \cdots + \mathcal{A}}_{h \text{ terms}}.$$

The $h$-tuple $(\mathcal{A}_1, \ldots, \mathcal{A}_h)$ is called an *additive system* of order $h$ if $\mathcal{A}_1 + \cdots + \mathcal{A}_h = \mathbf{N}_0$. The set $\mathcal{A}$ is an *additive basis*, or, simply, a *basis*, of order $h$ if $h\mathcal{A} = \mathbf{N}_0$. Note that every basis must contain 0 and 1. If $(\mathcal{A}_1, \ldots, \mathcal{A}_h)$ is an additive system of order $h$, then $\mathcal{A} = \bigcup_{i=1}^{h} \mathcal{A}_i$ is a basis of order $h$. The set $\mathcal{A}$ is an *asymptotic basis* of order $h$ if the sumset $h\mathcal{A}$ contains all sufficiently large integers.

For example, Lagrange's theorem states that the squares are a basis of order 4, and Wieferich's theorem states that the nonnegative cubes are a basis of order 9. Let $r_1, \ldots, r_h$ be relatively prime positive integers, and let

$$\mathcal{A}_i = r_i * \mathbf{N}_0 = \{r_i v_i : v_i \in \mathbf{N}_0\}$$

* Correspondence to: Department of Mathematics, Lehman College (CUNY), Bronx, NY 10468, United States.
*E-mail address:* melvyn.nathanson@lehman.cuny.edu.

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for $i = 1, \ldots, h$. There exists an integer $C$ such that every integer $n \geq C$ can be represented in the form $n = r_1 v_1 + \cdots + r_h v_h$ with $v_1, \ldots, v_h \in \mathbf{N}_0$ [8, Section 1.6], and so the set

$$\mathcal{A} = [0, C-1] \cup \bigcup_{i=1}^{h} \mathcal{A}_i$$

is an additive basis of order $h$.

Let $g \geq 2$. We can use the $g$-adic representation of the nonnegative integers to construct another class of additive bases of order $h$. Let $K_1, \ldots, K_h$ be pairwise disjoint subsets of $\mathbf{N}_0$ such that

$$\mathbf{N}_0 = K_1 \cup \cdots \cup K_h.$$

For $i = 1, \ldots, h$, let $\mathcal{F}(K_i)$ denote the set of all finite subsets of $K_i$, and let

$$\mathcal{A}_i = \left\{ \sum_{k \in F} \varepsilon_k g^k : F \in \mathcal{F}(K_i) \text{ and } \varepsilon_k \in \{0, 1, 2, \ldots, g-1\} \right\}. \tag{1}$$

Then $(\mathcal{A}_1, \ldots, \mathcal{A}_h)$ is an additive system of order $h$ with $\bigcap_{i=1}^{h} \mathcal{A}_i = \{0\}$, and $\mathcal{A} = \bigcup_{i=1}^{h} \mathcal{A}_i$ is a basis of order $h$.

The *counting function* of a set $\mathcal{A}$ of integers is

$$A(x) = \sum_{\substack{a \in \mathcal{A} \\ 1 \leq a \leq x}} 1.$$

If $\mathcal{A}$ is a basis of order $h$, then for all $x \in \mathbf{N}_0$ we have

$$x + 1 \leq \binom{A(x)+h}{h} \leq \frac{(A(x)+h)^h}{h!}$$

and so

$$\liminf_{x \to \infty} \frac{A(x)}{x^{1/h}} \geq (h!)^{1/h}.$$

The basis $\mathcal{A}$ of order $h$ is called a *thin basis* if

$$\limsup_{x \to \infty} \frac{A(x)}{x^{1/h}} < \infty.$$

One of the earliest general problems in additive number theory was that of the existence and construction of thin bases of finite order [11]. In 1937, Raikov [10] and Stöhr [14] independently solved this problem. Using the 2-adic representation and the partition $\mathbf{N}_0 = \bigcup_{i=1}^{h} K_i$, where $K_i = \{k \in \mathbf{N}_0 : k \equiv i-1 \pmod h\}$ for $i = 1, \ldots, h$, they proved that $\mathcal{A} = \bigcup_{i=1}^{h} \mathcal{A}_i$ is a thin basis of order $h$, where $\mathcal{A}_i$ is the set defined by (1) with $g = 2$. Many thin bases have been constructed using variations of the $g$-adic construction; see, for example, Cassels [2], Grekos [4], Nathanson [9] and Jia and Nathanson [6]. Recently, Blomer [1], Hofmeister [5], and Schmitt [12] constructed other examples of thin bases.

In 1940 Shatrovskiĭ [13] described a beautiful class of thin bases of order $h$ that depends on the linear diophantine equation $n = a_1 x_1 + \cdots + a_h x_h$ and does not use $g$-adic representations. This is an almost forgotten paper.[1] It has been mentioned in a few surveys of solved and unsolved problems in additive number theory (e.g. Erdős and Nathanson [3], Nathanson [7] and Stöhr [15]), but MathSciNet lists no publication that cites Shatrovskiĭ's paper. This paper contains an exposition of Shatrovskiĭ's construction of thin bases.

## 2. The construction

Fix an integer $h \geq 2$, and define

$$k_0 = k_0(h) = \left\lceil \frac{1}{2^{1/h} - 1} \right\rceil + 1. \tag{2}$$

Then

$$1 + \frac{1}{k} < 2^{1/h}$$

for all $k \geq k_0$.

[1] Shatrovskiĭ published his paper in Russian, with a summary in French. The transliteration of the author's name in the French summary is Chatrovsky, and this is the spelling that appears in *Mathematical Reviews*. I prefer the AMS and Library of Congress system for transliteration of Cyrillic into English.

Let $r_1, \ldots, r_h$ be a strictly increasing sequence of pairwise relatively prime positive integers, that is, $r_i < r_j$ and $(r_i, r_j) = 1$ for $1 \leq i < j \leq h$. Let $P$ be a positive integer such that

(i) $P \geq r_h - r_1$, and
(ii) if $1 \leq i < j \leq h$ and if the prime $p$ divides $r_j - r_i$, then $p$ divides $P$.

For every positive integer $k$ and for $i = 1, \ldots, h$, we define the integers

$$s_{i,k} = r_i + kP. \tag{3}$$

Then $s_{1,k} < s_{2,k} < \cdots < s_{h,k}$, and $s_{1,k} \geq s_{1,1} = r_1 + P \geq 2$.

Let $1 \leq i < j \leq h$. If a prime number $p$ divides both $s_{i,k}$ and $s_{j,k}$, then $p$ divides $s_{j,k} - s_{i,k} = r_j - r_i$ and so $p$ divides $P$. Therefore, $p$ divides both $r_i$ and $r_j$, which contradicts $(r_i, r_j) = 1$. It follows that $s_{1,k}, s_{2,k}, \ldots, s_{h,k}$ is a strictly increasing sequence of pairwise relatively prime positive integers. Moreover, if $1 \leq i < j \leq h$, then

$$\begin{aligned} 1 < \frac{s_{j,k}}{s_{i,k}} &= \frac{r_j + kP}{r_i + kP} \\ &= 1 + \frac{r_j - r_i}{r_1 + kP} \\ &\leq 1 + \frac{r_h - r_1}{r_1 + kP} \\ &\leq 1 + \frac{P}{r_1 + kP} \\ &< 1 + \frac{1}{k}. \end{aligned}$$

If $k \geq k_0$, then $s_{i,k} < s_{j,k} < 2^{1/h} s_{i,k}$.

For every positive integer $k$, we define the integer

$$S_k = \prod_{i=1}^{h} s_{i,k} = \prod_{i=1}^{h} (r_i + kP). \tag{4}$$

Then $s_{1,k}^h < S_k < s_{h,k}^h$, and $s_{h,k}^h < 2s_{1,k}^h < 2S_k$ for $k \geq k_0$.

For all $k \geq 1$ we have

$$\begin{aligned} 1 < \frac{S_{k+1}}{S_k} &= \prod_{i=1}^{h} \left( \frac{r_i + (k+1)P}{r_i + kP} \right) \\ &= \prod_{i=1}^{h} \left( 1 + \frac{P}{r_i + kP} \right) \\ &< \left( 1 + \frac{1}{k} \right)^h. \end{aligned}$$

If $k \geq k_0$, then

$$S_k < S_{k+1} < 2S_k.$$

For every positive integer $k$ and for $i = 1, \ldots, h$, we define the integers

$$a_{i,k} = \frac{S_k}{s_{i,k}} = \prod_{\substack{j=1 \\ j \neq i}}^{h} s_{j,k}. \tag{5}$$

Then $a_{1,k} > a_{2,k} > \cdots > a_{h,k}$. For $k \geq k_0$, we have

$$\frac{S_k}{a_{h,k}} = s_{h,k} < 2^{1/h} S_k^{1/h}. \tag{6}$$

Suppose that there is a prime number $p$ that divides $a_{i,k}$ for all $i = 1, \ldots, h$. Because $p$ divides $a_{1,k}$, it follows from (5) that $p$ divides $s_{\ell,k}$ for some $\ell \in \{2, 3, \ldots, h\}$. Because $p$ also divides $a_{\ell,k}$, it follows from (5) that $p$ divides $s_{m,k}$ for some $m \in \{1, 2, 3, \ldots, h\} \setminus \{\ell\}$. This is impossible because $(s_{\ell,k}, s_{m,k}) = 1$. Therefore, $a_{1,k}, a_{2,k}, \ldots, a_{h,k}$ is a strictly decreasing sequence of relatively prime positive integers.

**Lemma 1.** *Let $h \geq 2$, and let $r_1, \ldots, r_h$ be a strictly increasing sequence of pairwise relatively prime positive integers. For every positive integer $k$ and $i = 1, \ldots, h$, let $s_{i,k}$, $S_k$, and $a_{i,k}$ be the positive integers defined by (3)–(5), respectively. Let $k_1 \geq k_0(h)$. For every positive integer $t$, there is a unique nonnegative integer $\ell(t)$ defined by*

$$S_{k_1+\ell(t)} \leq S_{k_1}^t < S_{k_1+\ell(t)+1}. \tag{7}$$

*Then $\ell(1) = 0$ and the sequence $\{\ell(t)\}_{t=1}^{\infty}$ is strictly increasing. Moreover,*

$$\frac{1}{S_{k_1+\ell(t)}} < \frac{2}{S_{k_1}^t}. \tag{8}$$

**Proof.** Because the sequence $\{S_k\}_{k=1}^{\infty}$ is strictly increasing, for every positive integer $t$ there is a unique nonnegative integer $\ell(t)$ that satisfies inequality (7). In particular, $S_{k_1} \leq S_{k_1}^1 < S_{k_1+1}$ and so $\ell(1) = 0$. If $k \geq k_1 \geq k_0$, then $S_{k+1} < 2S_k$ and so

$$S_{k_1}^t < S_{k_1+\ell(t)+1} < 2S_{k_1+\ell(t)} \leq 2S_{k_1}^t < S_{k_1}^{t+1} < S_{k_1+\ell(t+1)+1}.$$

It follows that $\ell(t) < \ell(t+1)$.

Inequality (8) follows from

$$S_{k_1}^t < S_{k_1+\ell(t)+1} < 2S_{k_1+\ell(t)}.$$

This completes the proof. □

**Lemma 2.** *Let $h \geq 2$, and let $r_1, \ldots, r_h$ be a strictly increasing sequence of pairwise relatively prime positive integers. For every positive integer $k$ and $i = 1, \ldots, h$, let $s_{i,k}$, $S_k$, and $a_{i,k}$ be the positive integers defined by (3)–(5), respectively. For every integer $n$ such that $(h-1)S_k < n \leq L$, there exist positive integers $v_1, v_2, \ldots, v_h$ such that*

$$a_{1,k}v_1 + a_{2,k}v_2 + \cdots + a_{h,k}v_h = n$$

*and $1 \leq v_i \leq s_{i,k}$ for $i = 1, \ldots, h-1$ and $1 \leq v_h < L/a_{h,k}$.*

**Proof.** By Lemma 1, the integers $a_{1,k}, a_{2,k}, \ldots, a_{h,k}$ are relatively prime, and so there exist integers $u_1, u_2, \ldots, u_h$, not necessarily positive, such that

$$a_{1,k}u_1 + a_{2,k}u_2 + \cdots + a_{h,k}u_h = n.$$

For each $i = 1, 2, \ldots, h-1$ there is a unique integer $v_i \in \{1, 2, \ldots, s_{i,k}\}$ such that

$$v_i \equiv u_i \pmod{s_{i,k}}.$$

Then

$$\begin{aligned}
n &= \sum_{i=1}^{h} a_{i,k}u_i \\
&= \sum_{i=1}^{h-1} a_{i,k}v_i + \sum_{i=1}^{h-1} a_{i,k}(u_i - v_i) + a_{h,k}u_h \\
&= \sum_{i=1}^{h-1} a_{i,k}v_i + \sum_{i=1}^{h-1} a_{i,k}s_{i,k}\left(\frac{u_i - v_i}{s_{i,k}}\right) + a_{h,k}u_h \\
&= \sum_{i=1}^{h-1} a_{i,k}v_i + S_k \sum_{i=1}^{h-1}\left(\frac{u_i - v_i}{s_{i,k}}\right) + a_{h,k}u_h \\
&= \sum_{i=1}^{h-1} a_{i,k}v_i + a_{h,k}s_{h,k} \sum_{i=1}^{h-1}\left(\frac{u_i - v_i}{s_{i,k}}\right) + a_{h,k}u_h \\
&= \sum_{i=1}^{h} a_{i,k}v_i
\end{aligned}$$

where

$$v_h = s_{h,k} \sum_{i=1}^{h-1}\left(\frac{u_i - v_i}{s_{i,k}}\right) + u_h \in \mathbf{Z}.$$

The inequality

$$\sum_{i=1}^{h-1} a_{i,k} v_i \leq \sum_{i=1}^{h-1} a_{i,k} s_{i,k} = (h-1)S_k$$

implies that if $n > (h-1)S_k$, then

$$a_{h,k} v_h = n - \sum_{i=1}^{h-1} a_{i,k} v_i \geq n - (h-1)S_k > 0$$

and so $v_h \geq 1$. Moreover, $a_{h,k} v_h < n \leq L$ and so $v_h < L/a_{h,k}$. This completes the proof. □

**Theorem 1.** *Let $h \geq 2$, and let $r_1, \ldots, r_h$ be a strictly increasing sequence of pairwise relatively prime positive integers. For every positive integer $k$ and $i = 1, \ldots, h$, let $s_{i,k}$, $S_k$, and $a_{i,k}$ be the positive integers defined by (3)–(5), respectively. Let $k_1 \geq k_0(h)$. For every positive integer $t$, let $\ell(t)$ be the integer defined by (7), and let*

$$V_t = \left\{ a_{i,k_1+\ell(t)} v_i : v_i \in \left[1, s_{i,k_1+\ell(t)}\right] \text{ for } i \in [1, h-1] \text{ and } v_h \in \left[1, \frac{(h-1)S_{k_1+\ell(t+1)}}{a_{h,k_1+\ell(t)}}\right] \right\}.$$

*Then:*

(i)

$$|V_t| < \frac{2(h-1)S_{k_1+\ell(t+1)}}{a_{h,k_1+\ell(t)}}.$$

(ii)

$$hV_t \supseteq \left[(h-1)S_{k_1+\ell(t)} + 1, (h-1)S_{k_1+\ell(t+1)}\right].$$

(iii) *For every positive integer $t_1$, the set*

$$\mathcal{A}_{t_1} = \bigcup_{t=t_1}^{\infty} V_t$$

*is an asymptotic basis of order $h$.*

(iv) *The set*

$$\mathcal{A} = [0, S_{k_1}] \cup \bigcup_{t=1}^{\infty} V_t$$

*is a basis of order $h$.*

**Proof.** For $i = 1, \ldots, h-1$ we have

$$s_{i,k_1+\ell(t)} = \frac{S_{k_1+\ell(t)}}{a_{i,k_1+\ell(t)}} < \frac{S_{k_1+\ell(t+1)}}{a_{h,k_1+\ell(t)}}$$

and so

$$\begin{aligned} |V_t| &\leq \sum_{i=1}^{h-1} s_{i,k_1+\ell(t)} + \frac{(h-1)S_{k_1+\ell(t+1)}}{a_{h,k_1+\ell(t)}} \\ &< \frac{2(h-1)S_{k_1+\ell(t+1)}}{a_{h,k_1+\ell(t)}}. \end{aligned}$$

This proves (i).

Lemma 2 implies statement (ii), and statement (ii) implies that the $h$-fold sumset $h\mathcal{A}_{t_1}$ contains every integer $n \geq (h-1)S_{k_1+\ell(t_1)} + 1$. Thus, $\mathcal{A}_{t_1}$ is an asymptotic basis of order $h$. In particular, $h\mathcal{A}_1$ contains every integer $n \geq (h-1)S_{k_1} + 1$. Because

$$[0, (h-1)S_{k_1}] \subseteq [0, hS_{k_1}] = h[0, S_{k_1}]$$

it follows that $\mathcal{A} = [0, S_{k_1}] \cup \mathcal{A}_1$ is a basis of order $h$. This completes the proof. □

**Theorem 2.** *The set*

$$\mathcal{A} = [0, S_{k_1}] \cup \bigcup_{t=1}^{\infty} V_t$$

*constructed in Theorem 1 is a thin basis of order $h$.*

**Proof.** Let $A(x)$ be the counting function of the set $\mathcal{A}$. For all $x \geq S_{k_1}$, there is a smallest positive integer $t^*$ such that $S_{k_1+\ell(t^*)} \leq x$. Then

$$S_{k_1+\ell(t^*)} \leq x < S_{k_1+\ell(t^*+1)} \leq S_{k_1}^{t^*+1}$$

and

$$\frac{1}{S_{k_1}^{t^*+1}} < \frac{1}{x}. \tag{9}$$

Because $V_t(x) \leq |V_t|$ for all $x$, we have

$$A(x) \leq S_{k_1} + \sum_{t=1}^{\infty} V_t(x) \leq S_{k_1} + \sum_{t=1}^{t^*} |V_t| + \sum_{t=t^*+1}^{\infty} V_t(x).$$

We shall estimate the two sums separately. From Theorem 1 (i) and inequality (6), we obtain

$$\begin{aligned}
\sum_{t=1}^{t^*} |V_t| &\leq \sum_{t=1}^{t^*} \frac{2(h-1)S_{k_1+\ell(t+1)}}{a_{h,k_1+\ell(t)}} \\
&\leq 2(h-1) \sum_{t=1}^{t^*} \frac{S_{k_1}^{t+1}}{a_{h,k_1+\ell(t)}} \\
&= 2(h-1)S_{k_1} \sum_{t=1}^{t^*} \frac{S_{k_1}^{t}}{a_{h,k_1+\ell(t)}} \\
&< 2(h-1)S_{k_1} \sum_{t=1}^{t^*} \frac{S_{k_1+\ell(t)+1}}{a_{h,k_1+\ell(t)}} \\
&\leq 4(h-1)S_{k_1} \sum_{t=1}^{t^*} \frac{S_{k_1+\ell(t)}}{a_{h,k_1+\ell(t)}} \\
&< 2^{2+1/h}(h-1)S_{k_1} \sum_{t=1}^{t^*} S_{k_1+\ell(t)}^{1/h} \\
&\leq 2^{2+1/h}(h-1)S_{k_1} \sum_{t=1}^{t^*} S_{k_1}^{t/h} \\
&< \left( \frac{2^{1+1/h}h(h-1)S_{k_1}^{1+1/h}}{S_{k_1}^{1/h}-1} \right) S_{k_1}^{t^*/h} \\
&< \left( \frac{2^{2+1/h}(h-1)S_{k_1}^{1+1/h}}{S_{k_1}^{1/h}-1} \right) S_{k_1+\ell(t^*)+1}^{1/h} \\
&< \left( \frac{2^{2+2/h}(h-1)S_{k_1}^{1+1/h}}{S_{k_1}^{1/h}-1} \right) S_{k_1+\ell(t^*)}^{1/h} \\
&\leq \left( \frac{2^{2+2/h}(h-1)S_{k_1}^{1+1/h}}{S_{k_1}^{1/h}-1} \right) x^{1/h}.
\end{aligned}$$

To estimate the second sum, we observe that if $a_{i,k_1+\ell(t)}v_i \in V_t$ and $a_{i,k_1+\ell(t)}v_i \leq x$, then

$$1 \leq v_i \leq \frac{x}{a_{i,k_1+\ell(t)}} \leq \frac{x}{a_{h,k_1+\ell(t)}}.$$

Applying inequalities (6), (8) and (9), we obtain

$$\begin{aligned}
\sum_{t=t^*+1}^{\infty} V_t(x) &\leq \sum_{t=t^*+1}^{\infty} \sum_{i=1}^{h} \frac{x}{a_{i,k_1+\ell(t)}} \\
&< hx \sum_{t=t^*+1}^{\infty} \frac{1}{a_{h,k_1+\ell(t)}}
\end{aligned}$$

$$
\begin{aligned}
&< 2^{1/h} hx \sum_{t=t^*+1}^{\infty} \left( \frac{1}{S_{k_1+\ell(t)}} \right)^{1-1/h} \\
&< 2^{1/h} hx \sum_{t=t^*+1}^{\infty} \left( \frac{2}{S_{k_1+\ell(t)+1}} \right)^{1-1/h} \\
&< 2hx \sum_{t=t^*+1}^{\infty} \frac{1}{S_{k_1}^{(1-1/h)t}} \\
&= \left( \frac{2hS_{k_1}^{1-1/h}}{S_{k_1}^{1-1/h} - 1} \right) \frac{x}{S_{k_1}^{(1-1/h)(t^*+1)}} \\
&< \left( \frac{2hS_{k_1}^{1-1/h}}{S_{k_1}^{1-1/h} - 1} \right) \frac{x}{x^{1-1/h}} \\
&= \left( \frac{2hS_{k_1}^{1-1/h}}{S_{k_1}^{1-1/h} - 1} \right) x^{1/h}.
\end{aligned}
$$

This completes the proof. □

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