
Subalgebras of a Polynomial Ring That Are Not Finitely Generated

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Abstract. Let R_1 be a commutative ring, let R_2 be a finitely generated extension ring of R_1 , and let S be a ring that is intermediate between R_1 and R_2 . For $R_1 = R[x]$ and $R_2 = R[x, y]$, there are simple combinatorial constructions of intermediate rings S that are not finitely generated over $R[x]$.

Let R_1 and R_2 be commutative rings with $R_1 \subseteq R_2$. The ring R_2 is *finitely generated as a ring* over R_1 if there is a finite subset \mathcal{X} of R_2 such that every element of R_2 can be represented as a linear combination of monomials in \mathcal{X} with coefficients in R_1 . The ring R_2 is *finitely generated as a module* over R_1 if there is a finite subset \mathcal{X} of R_2 such that every element of R_2 can be represented as a linear combination of elements of \mathcal{X} with coefficients in R_1 .

Let R_2 be finitely generated as a ring over R_1 . By Hilbert's basis theorem, if R_1 is Noetherian, then R_2 is also Noetherian. Let S be a ring that is *intermediate* between R_1 and R_2 , that is,

$$R_1 \subseteq S \subseteq R_2.$$

Artin and Tate [1] proved that if R_1 is Noetherian and if R_2 is finitely generated as a module over S , then S is finitely generated as a ring over R_1 . They used this to prove Hilbert's Nullstellensatz (cf. Zariski [3], Kunz [2, Lemma 3.3]).

It is natural to ask: If R_2 is finitely generated as a ring over R_1 and if the ring S is intermediate between R_1 and R_2 , then is S finitely generated as a ring over R_1 ? The answer is "no," and the purpose of this note is to give simple combinatorial constructions of intermediate rings S that are not finitely generated over R_1 .

Let \mathbf{N} denote the set of positive integers and \mathbf{N}_0 the set of nonnegative integers.

Theorem 1. *Let λ be a positive real number or $\lambda = \infty$. Let Λ be a subset of $\mathbf{N} \times \mathbf{N}_0$ with $(1, 0) \in \Lambda$ such that*

$$\sup \left(\frac{b}{a} : (a, b) \in \Lambda \right) = \lambda \tag{1}$$

and

$$\frac{b}{a} < \lambda \quad \text{for all } (a, b) \in \Lambda. \tag{2}$$

Consider the set of monomials

$$M(\Lambda) = \{x^a y^b : (a, b) \in \Lambda\}.$$

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Let R be a commutative ring, and let $R[M(\Lambda)]$ be the subring of $R[x, y]$ generated by $M(\Lambda)$. Then

$$R[x] \subseteq R[M(\Lambda)] \subseteq R[x, y]$$

and $R[M(\Lambda)]$ is not finitely generated as a ring over $R[x]$.

For example, the set $\Lambda_1 = \{(1, n) : n \in \mathbf{N}_0\}$ satisfies conditions (1) and (2) with $\lambda = \infty$. The corresponding set of monomials is

$$M(\Lambda_1) = \{x, xy, xy^2, xy^3, \dots\},$$

and the ring

$$R[M(\Lambda_1)] = R[x, xy, xy^2, xy^3, \dots]$$

is intermediate between $R[x]$ and $R[x, y]$. Similarly, if $(f_n)_{n=-1}^{\infty}$ is the sequence of Fibonacci numbers with $f_{-1} = 1$, $f_0 = 0$, and $f_1 = 1$, then the set $\Lambda_2 = \{(f_{2n-1}, f_{2n}) : n \in \mathbf{N}_0\}$ satisfies conditions (1) and (2) with $\lambda = (\sqrt{5} + 1)/2$. By Theorem 1, the intermediate rings $R[M(\Lambda_1)]$ and $R[M(\Lambda_2)]$ are not finitely generated over $R[x]$.

Note that inequalities (1) and (2) imply that the sets Λ and $M(\Lambda)$ are infinite. If $\lambda = \infty$, then (1) implies (2).

Proof. Because $(1, 0) \in \Lambda$, we have $x \in M(\Lambda)$ and $R[x] \subseteq R[M(\Lambda)] \subseteq R[x, y]$.

Let \mathcal{F} be a finite subset of $R[M(\Lambda)]$. For every polynomial f in \mathcal{F} , there is a finite set $M^*(f)$ of monomials in $M(\Lambda)$ such that f is a linear combination of products of monomials in $M^*(f)$. This set of monomials is not necessarily unique (for example, $(xy)(xy^4) = (xy^2)(xy^3)$ in $R[M(\Lambda_1)]$), but we choose, for each polynomial f in \mathcal{F} , one set $M^*(f)$ of monomials in $M(\Lambda)$ that generates f . Because \mathcal{F} is a finite set of polynomials, the set

$$M^*(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} M^*(f)$$

is a finite set of monomials in $M(\Lambda)$. Moreover, $f \in R[M^*(\mathcal{F})]$ for all $f \in \mathcal{F}$, and so

$$R[\mathcal{F}] \subseteq R[M^*(\mathcal{F})] \subseteq R[M(\Lambda)].$$

We shall prove that $R[M^*(\mathcal{F})] \neq R[M(\Lambda)]$.

Let

$$\beta = \max \left(\frac{b}{a} : x^a y^b \in M^*(\mathcal{F}) \right).$$

Applying inequality (2) to the finite set $M^*(\mathcal{F})$, we obtain $\beta < \lambda$. If $(A, B) \in \mathbf{N} \times \mathbf{N}_0$ and $x^A y^B \in R[M^*(\mathcal{F})]$, then $x^A y^B$ is an R -linear combination of products of monomials in $M^*(\mathcal{F})$. This implies that $x^A y^B$ is a product of monomials in $M^*(\mathcal{F})$. Thus, there is a finite sequence $((a_i, b_i))_{i=1}^n$ of ordered pairs in Λ such that $x^{a_i} y^{b_i} \in M^*(\mathcal{F})$ for all $i = 1, \dots, n$ and

$$x^A y^B = \prod_{i=1}^n x^{a_i} y^{b_i} = x^{\sum_{i=1}^n a_i} y^{\sum_{i=1}^n b_i}.$$

The inequality $b_i \leq \beta a_i$ for $i = 1, \dots, n$ implies that

$$\frac{B}{A} = \frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n a_i} \leq \frac{\beta \sum_{i=1}^n a_i}{\sum_{i=1}^n a_i} = \beta.$$

Condition (1) implies that the ring $R[M(\Lambda)]$ contains monomials $x^A y^B$ with $\beta < B/A < \lambda$. It follows that $x^A y^B \notin R[M^*(\mathcal{F})]$ and so $R[M^*(\mathcal{F})] \neq R[M(\Lambda)]$. Therefore, $R[\mathcal{F}] \neq R[M(\Lambda)]$, and the ring $R[M(\Lambda)]$ is not finitely generated. This completes the proof. ■

Theorem 1 suggests the following problems.

1. Classify the sets $M(\Lambda)$ of monomials of the form $x^a y^b$ such that

$$R[x] \subseteq R[M(\Lambda)] \subseteq R[x, y] \quad (3)$$

and the ring $R[M(\Lambda)]$ is not finitely generated over $R[x]$.

2. More generally, describe all rings S that are intermediate between $R[x]$ and $R[x, y]$ and are not finitely generated over R .

Added in proof. Ali Cherachi and Sebastián Herrero (independently) have proved that Theorem 1 gives not only a sufficient condition but also a necessary condition for a subalgebra of the form $R[M]$ satisfying (3) not to be finitely generated. This solves problem 1.

Herrero and a referee observed that examples of not finitely generated subalgebras of a polynomial ring have appeared on Math Overflow (<http://mathoverflow.net/questions/48798/non-finitely-generated-sub-algebra-of-a-finitely-generated-algebra>).

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